CS 70 Discrete Mathematics and Probability Theory
Fall 2024 Hug, Rao

Final Solutions

PRINT Your Name: Oski Bear

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1. Pledge.

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2. Propositions and Proofs

1. In the following parts, let P, Q and R be propositions. Determine the truth value of the following statements as always true, always false, or either true or false.

   (a) $P \vee Q \vee R \vee \neg R$
   Answer: $R \vee \neg R$ is true, so taking the OR with anything else still makes the expression always true.

   (b) $(\neg P \implies (R \wedge \neg R)) \implies P$
   Answer: True. This is basically a proof by contradiction for the proposition P.

   (c) $(R \wedge \neg R) \vee (Q \wedge P)$
   Answer: Could be either. The $R \wedge \neg R$ evaluates to False, so the truth value depends entirely on $Q \wedge P$, which could be anything.

2. Let $P(x)$, $Q(x)$, and $R(x,y)$ be predicates for $x, y \in U$, where U is a non-empty universe.
   Suppose we know the following is true:

$$(\forall x \in U)\big(P(x) \implies [Q(x) \vee (\exists y \in U)(R(x,y))]\big)$$

   (a) Consider an element $x \in U$ such that $P(x)$ is True, and $(\forall y \in U)(\neg R(x,y))$ is True. What is $Q(x)$?
   Answer: Always true. If $P(x)$ is true, the implication tells us that $Q(x) \vee (\exists y \in U)(R(x,y))$ is also true. However, since $R(x,y)$ is always false, it must be the case that $Q(x)$ is true.

   (b) Consider an element $x \in U$ such that $Q(x)$ is False, and $(\forall y \in U)(\neg R(x,y))$ is True. What is $P(x)$?
   Answer: Always false. If $P(x)$ is always false, the implication is vacuously true. However, for cases where $P(x)$ is true, the implication tells us that $Q(x) \vee (\exists y \in U)(R(x,y))$ is also true. However, since $Q(x)$ is false and $R(x,y)$ is also always false, this cannot happen.

   (c) Consider an element $x \in U$ such that $P(x)$ is True, and $(\exists y \in U)(R(x,y))$ is True. What is $Q(x)$?
   Answer: Either true or false. If $P(x)$ is true, then the implication tells us that $Q(x) \vee (\exists y \in U)(R(x,y))$ is also true. However, we know that $R(x,y)$ is true for some value of y: this immediately makes the conclusion true, so we cannot say anything about $Q(x)$; it could be either true or false.

3. Fill in the following blank using an expression involving *only* $\max(a,b)$ and $\min(a,b)$:

$$|a-b| = \_\_\_\_\_\_\_\_\_\_.$$

Answer: $\max(a,b)-\min(a,b)$. Consider $a \geq b$. Then we get $a = \max(a,b)$ and $b = \min(a,b)$ and $a-b$ is positive and thus is $|a-b|$. Similarly, when $a < b$. Then we get $b = \max(a,b)$ and $a = \min(a,b)$ and $b-a$ is positive and thus is $|a-b|$.

3. Induction

1. (0 points) Dogs know induction.
2. Consider a predicate $P(n)$ for $n \in \mathbb{N}$. Fill in the blank to make the following statement true.

$$(\forall n \in \mathbb{N})(P(n) \implies P(n+1)) \equiv \neg(\exists n \in \mathbb{N})(P(n) \wedge \_\_\_\_)$$

Answer: $\neg P(n+1)$. It says there does not exist an n where $\neg(P(n) \implies P(n+1)) \equiv P(n) \wedge \neg P(n+1)$

3. Recall the Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, and $\forall i \geq 2$, $F_i = F_{i-1} + F_{i-2}$. We would like to show that $F_i \geq c^{i-1}$ for some specific value of c and all $i \geq 1$.

(a) What is the value of c? Your value of c should be as large as possible. (Note: $F_1 \geq c^0$ for any c.)
Answer: In the base case, we have $F_1 \geq c^0$ for any c. If $c \geq 1$, then we can say that

$$F_i = F_{i-1} + F_{i-2} \geq c^{i-2} + c^{i-3},$$

which we want to be $\geq c^{i-1}$. This gives us the equation

$$c^{i-2} + c^{i-3} \geq c^{i-1} \implies 1 + c \geq c^2 \implies -c^2 + c + 1 \geq 0.$$

The solutions to this quadratic equation are $\frac{1}{2}(1 \pm \sqrt{5})$, so taking the larger solution, we have $c = \frac{1}{2}(1+\sqrt{5})$.

(b) (8 points) Prove that your answer is correct.
Answer: See above.

4. Stable Matching

1. Consider the following preferences lists for a stable matching instance:

Jobs	Preferences
A	$1 > 2$
B	$2 > 1$

Candidates	Preferences
1	$B > A$
2	$A > B$

(a) Provide a job optimal stable pairing.
Answer: $(A,1),(B,2)$. The jobs can't do any better which makes it stable and optimal.

(b) Provide a candidate optimal stable pairing.
Answer: $(A,2),(B,1)$. The candidates can't do any better which makes it stable and optimal.

2. Consider the following preferences lists for a stable matching instance:

Jobs	Preferences
A	$1 > 2 > 3 > 4$
B	$2 > 1 > 4 > 3$
C	$3 > 4 > 1 > 2$
D	$4 > 3 > 2 > 1$

Candidates	Preferences
1	$B > A > C > D$
2	$A > B > D > C$
3	$D > C > A > B$
4	$C > D > B > A$

(a) How many stable matchings are there for these preference lists?
Answer: 4. There is no stable matching where jobs $\{A,B\}$ are paired with any of candidates $\{3,4\}$, and similarly there is no stable matching where jobs $\{C,D\}$ are paired with any of candidates $\{1,2\}$. That is, jobs $\{A,B\}$ must be paired with one of candidates $\{1,2\}$, and jobs $\{C,D\}$ must be paired with one of candidates $\{3,4\}$. As such, there are a total of $2 \times 2 = 4$ stable matchings.

(b) Provide a stable matching that is neither candidate optimal nor job optimal.
Answer: $\{(A,1),(B,2),(C,4),(D,3)\}$ OR $\{(A,2),(B,1),(C,3),(D,4)\}$.
We can combine the job optimal partners for A and B (i.e. 1 and 2 respectively) and the candidate optimal partners for 3 and 4 (i.e. D and C respectively) to create a stable matching that is neither job optimal nor candidate optimal.
Alternatively, we can combine the job optimal partners for C and D (i.e. 3 and 4, respectively) and the candidate optimal partners for 1 and 2 (i.e. B and A, respectively).

5. Graphs

All graphs are simple and undirected unless otherwise specified.

1. Let $G = (V,E)$, with k **connected components**.

   (a) What is the sum of the degrees of all vertices in G? (Your answer should possibly be in terms of $|E|$, $|V|$ and/or k.)
   Answer: $2|E|$. By the handshaking lemma, the sum of the degrees of all vertices in a graph is $2|E|$, since each edge is counted twice.

   (b) What is the maximum possible number of edges in G? (Give a tight bound in terms $|V|$ and k. Your answer should not include $|E|$.)
   Answer: $\binom{|V|-k+1}{2}$. To maximize the number of edges in the graph, we should have a single large connected component, with the rest of the components as isolated vertices. Here, the single large connected component would have $|V|-(k-1)$ vertices (as a clique, with all possible edges present), and we have $k-1$ remaining isolated vertices.

   (c) What is the minimum possible number of edges in G? (Give a tight bound in terms of $|V|$ and k. Your answer should not include $|E|$.)
   Answer: $|V|-k$. To minimize the number of edges in the graph, we want as many isolated vertices as possible; we should only ever add edges to ensure that we actually have k connected components. This means that we should have $k-1$ isolated vertices, and a single connected component with $|V|-(k-1)$ vertices and $|V|-(k-1)-1 = |V|-k$ edges to connect them, forming a tree.

2. Consider a hypercube $G = (V,E)$.

   (a) Recall a vertex coloring is an assignment of colors to vertices such that the endpoints of every edge are colored differently. What is the minimum number of colors needed to vertex color G? (Possibly in terms of $|V|$ and $|E|$.)
   Answer: 2. A hypercube is bipartite and thus it can be 2 colored.

   (b) Recall an edge coloring is an assignment of colors to edges such that any two edges incident to the same vertex are colored differently. What is the minimum number of colors needed to edge color G? (Possibly in terms of $|V|$ and $|E|$.)
   Answer: $\log_2 |V|$. The hypercube of dimension d has $|V| = 2^d$, and thus $d = \log_2 |V|$. The edges corresponding to each dimension do not share any endpoints and thus can all be colored with one color. This means that we need $d = \log_2 |V|$ colors in total.

3. Removing the edges in a cycle of length k from a connected graph produces a graph with at most _______ connected components. (Give a tight bound.)

   Answer: k. If we remove the cycle of length k, suppose we end up with c components. Now, suppose we "collapse" each resulting component down into a single vertex (we do this, because we're really only interested in the edges *across* components, not within a single component). Notice that the cycle that we just removed must have passed through each one of these components; at minimum, this would take c edges to form a cycle through these collapsed vertices. Maximizing c, since the cycle is of length k, this means that we must have at most k connected components.

   Alternatively, we can see that the upper bound must be at least k by considering a graph that is just a k length cycle. We now need to show that k is the smallest upper bound. Assume for the sake of contradiction that we have $k+1$ connected components after removing the edges. When we add back an edge, it can decrease the number of connected components by at most 1. However, the last edge added back completes the cycle and thus cannot decrease the number of connected components, thus we would end up at 2 connected components, which is a contradiction.

6. Modular Arithmetic

In the following parts, when working under arithmetic modulo N, your answers should be given in the range $\{0,1,\ldots,N-1\}$.

1. What is $2^{63} \pmod{77}$?

   Answer: $8 \pmod{77}$. $2^{(11-1)(7-1)} \equiv 2^{60} \equiv 1 \pmod{77}$ as $2^{(p-1)(q-1)} \equiv 1 \pmod{pq}$ by RSA.

2. What is $1 \times 2 \times 3 \times 4 \times 5 \times 6 \pmod{7}$?

   Answer: $-1 = 6 \pmod{7}$. Every element has an inverse and thus pair up to yield 1, except for 1 and -1 which are their own inverses, and multiplying them together yields -1.

3. Compute the multiplicative inverse of $63 \pmod{71}$.

   Answer: $-9 \equiv 62 \pmod{71}$. Using the iterative version of EGCD, we have

$$\begin{aligned}
63(0)+71(1) &= 71 && (E_1)\\
63(1)+71(0) &= 63 && (E_2)\\
63(-1)+71(1) &= 8 && (E_3 = E_1 - E_2)\\
63(8)+71(-7) &= 7 && (E_4 = E_2 - 7E_3)\\
63(-9)+71(6) &= 1 && (E_5 = E_3 - E_4)
\end{aligned}$$

   This leaves us with $63(-9) \equiv 1 \pmod{71}$, so the inverse is $-9 \equiv 62 \pmod{71}$.

4. Consider equations of the form $ax \equiv b \pmod{pq}$, for distinct primes p and q. Note that there are $(pq)^2$ possible values of (a,b) pairs modulo pq.

   (a) How many values $a \in \{0,\ldots,pq-1\}$ are relatively prime to p?

   Answer: $(p-1)q$. There are q multiples of p in the set, so the remaining $pq-q=(p-1)q$ values are relatively prime to p.

   (b) How many values $a \in \{0,\ldots,pq-1\}$ are relatively prime to p and q?

   Answer: $pq-p-q+1=(p-1)(q-1)$. There are q multiples of p and p multiples of q; however, we cannot just take $pq-p-q$ as our answer, since we've counted 0 twice (as it's a multiple of both p and q). Adding 1 to account for this, we have a total of $pq-p-q+1=(p-1)(q-1)$ values relatively prime to both p and q.

(c) For how many pairs (a,b) does $ax \equiv b \pmod{pq}$ have a unique solution?

Answer: $(p-1)(q-1)pq$. For a with $\gcd(a,pq) = 1$, the equivalence has a solution for all pq values of b, since a is always invertible. This gives $(p-1)(q-1)pq$ different pairs; we have $(p-1)(q-1)$ possibilities for a, and pq possibilities for b.

(d) Consider the equation $ax \equiv b \pmod{pq}$ where $\gcd(a,pq) = p$. If x is a solution to the equivalence, then $x + \_\_\_\_\_$ is also a solution.

Answer: q. $a(x+q) = ax + aq = ax + kpq \equiv ax \equiv b \pmod{pq}$. Here, the second equality is due to the fact that $p \mid a$.

(e) For how many pairs (a,b) does $ax \equiv b \pmod{pq}$ have exactly p solutions?

Answer: $(q-1)q$. Firstly, a cannot be zero: if it was, we'd either have 0 solutions (if $b \neq 0$) or any x would work (if $b = 0$).

There are two cases; either a is a multiple of p, or a is not a multiple of p. If a is a multiple of p, then we can write $a = kp$ for some $k \in \{1,\ldots,q-1\}$. Here, there can only be a solution when b is also a multiple of p; otherwise, the LHS is a multiple of p while the RHS is not, resulting in 0 solutions to the equivalence. So, taking b as a multiple of p as well, we can write $b = k'p$ for $k' \in \{0,1,\ldots,q-1\}$. This means that we can simplify

$$\begin{aligned} ax &\equiv b \pmod{pq} \\ kpx &\equiv k'p \pmod{pq} \\ kpx &= k'p + npq && (n \in \mathbb{Z}) \\ kx &= k' + nq \\ kx &\equiv k' \pmod{q} \end{aligned}$$

This equation has exactly one solution modulo q, since $0 < k < q$ and thus k is always invertible. Further, any solution x to this equivalence modulo q generates p different solutions for the original equivalence of the form $x + iq$ for $i \in \{0,1,\ldots,p-1\}$ (justified in part (d)), which is exactly what we're looking for.

Since there are $q-1$ possible choices for a, and q possible choices for b, this gives us $(q-1)q$ pairs with exactly p solutions.

On the other hand, if a is not a multiple of p, we claim that it is impossible for the equivalence to have p solutions. Here, we'll look at two more cases: either a is a multiple of q, or a is not a multiple of q.

If a is not a multiple of p but is a multiple of q, the same reasoning from earlier (when considering a as a multiple of p) holds for q; we end up concluding that there are q different solutions to the equivalence, which is not what we're looking for.

If a is neither a multiple of p nor a multiple of q, then $\gcd(a,pq) = 1$, and from part (c) there is only one unique solution to the equivalence, which is also not what we're looking for.

This means that we can only have p different solutions when a and b are both multiples of p (with a nonzero), giving us our answer of $(q-1)q$.

(f) (5 points) For how many pairs (a,b) does $ax \equiv b \pmod{pq}$ have 0 solutions?

Answer: $(q-1)(pq-q) + (p-1)(pq-p) + (pq-1)$. There are three cases.

If a is a nonzero multiple of p, then there is no solution if b is not a multiple of p. There are $q-1$ possibilities for a, and $pq-q$ possibilities for b, making $(q-1)(pq-q)$ possibilities in this case.

If a is a nonzero multiple of q, then there is no solution if b is not a multiple of q. There are $p-1$ possibilities for a, and $pq-p$ possibilities for b, making $(p-1)(pq-p)$ possibilities in this case.

Lastly, if $a \equiv 0$, then there is no solution if $b \not\equiv 0$. There are $pq-1$ possibilities for b in this case.

Alternatively, notice that the number of solutions must be one of $\{0,1,p,q,pq\}$. We've already computed the number of pairs that give $\{1, p, q\}$ solutions in prior parts:

- From part (c), there are $(p-1)(q-1)pq$ pairs with exactly 1 solution.
- From part (e), there are $(q-1)q$ pairs with exactly p solutions.
- Extending part (e) by swapping p and q, there are $(p-1)p$ pairs with exactly q solutions.

Additionally, there is exactly one pair $a = b = 0$ with exactly pq solutions. Since these are all of the pairs that we want to *exclude*, there are a total of

$$(pq)^2 - (p-1)(q-1)pq - (q-1)q - (p-1)p - 1$$

pairs with a unique solution, which is an equivalent expression.

5. For any natural number $N \geq 1$, if $(\forall a, b \in \{1, \ldots, N-1\})(ab \not\equiv \_\_\_\_ \pmod{N})$, then N is prime. (Fill in the blank to make the statement true.)

   Answer: 0. If it is prime, every a has an inverse and thus, $ab \neq 0$ unless $b = 0$. If it is composite, i.e, $N = xy$, thus taking $a = x$ and $b = y$ gives an example where $ab = 0 \pmod{N}$.

6. (5 points) Recall the CRT implies that there is a unique $x \pmod{pq}$ where $x \equiv a \pmod{p}$ and $x \equiv b \pmod{q}$, for distinct primes p and q. Use this fact to prove that there are $(p-1)(q-1)$ values in $\{0, \ldots, pq-1\}$ that are relatively prime to pq.

   Answer: If a and b are both not zero, then x cannot be a multiple of p or q, and thus is relatively prime to pq. There are $(p-1)(q-1)$ values of a and b pairs, and each pair gives us a unique x in $\pmod{pq}$ by CRT.

7. Polynomials

1. What is $\Delta_0(x)$ for the points $(0,0)$, $(1,1)$, $(4,1)$ under arithmetic modulo 5?

   Answer: $-x^2 + 5x - 4 \pmod{5}$. $(x-1)(x-4)((-1)(-4))^{-1} = -x^2 + 5x - 4 \pmod{5}$

2. Suppose we would like to perform the Berlekamp–Welch algorithm, where a degree 0 polynomial (for a message of size $n = 1$) was sent and $k = 1$ error is possible.

   Working in GF(5), the points $(0,1)$, $(2,1)$, $(3,4)$ are received.

   (a) What is the original polynomial?

   Answer: $P(x) = 1$. This is the unique polynomial that fits $n + k = 1 + 1 = 2$ of the points.

   (b) Let $Q(x) = q_1 x + q_0$ and $E(x) = x + b_0$. Write out the equation in the Berlekamp–Welch system for the point $(3,4)$.

   Answer: $3q_1 + q_0 = 4(3 + b_0)$. Plug in $i = 3$ and $r_i = 4$ to $Q(i) = r_i E(i)$.

3. Consider the $\Delta_i(x)$ polynomials for the points $(0, y_0), (1, y_1), \ldots, (n-1, y_{n-1})$, modulo a prime p.

   (a) What is the value of $\Delta_i(i)$? (Your answer should either be a numerical value or an expression possibly in terms of n and p.)

   Answer: 1. $\Delta_i(x) = \Pi_{j \neq i}(x - j)(\Pi_{j \neq i}(i - j))^{-1}$ thus $\Delta_i(i) = 1$.

   (b) What is $\Delta_i(j)$ for $j \neq i$? (Your answer should either be a numerical value or an expression possibly in terms of n and p.)

   Answer: 0. $\Delta_i(x) = \Pi_{j \neq i}(x - j)(\Pi_{j \neq i}(i - j))^{-1}$ thus $\Delta_i(j) = 0$ due to the factor of $(j - j)$.

   (c) Consider the points $(0, y_0), (1, y_1)$ and $(2, y_2)$. What is the corresponding degree 2 polynomial $P(x)$ in terms of the delta polynomials $\Delta_0(x)$, $\Delta_1(x)$, and $\Delta_2(x)$?

   Answer: $y_0 \Delta_0(x) + y_1 \Delta_1(x) + y_2 \Delta_2(x)$ by Lagrange interpolation.

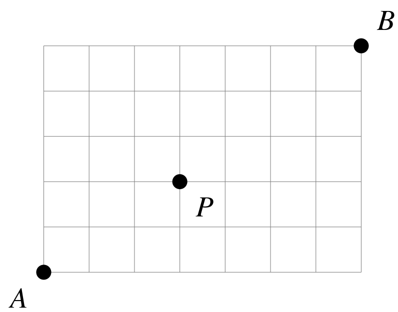
8. Counting Basics

Throughout this question, you may leave your answers unsimplified (i.e. you can leave binomial coefficients, factorials, exponents, etc. as is), but you should not use any summation or product notation (i.e. you may not use $\sum$ or $\prod$).

1. How many distinct ways can you rearrange the letters in the word "BERKELEY"?
   Answer: $\frac{8!}{3!}$. We have 8 letters, but 3 duplicate E's; there are 8! ways of arranging the 8 letters, and we must divide out the 3! ways of overcounting arrangements of the E's.
2. How many ways can you arrange n labeled balls into m labeled bins?
   Answer: m^n. We have m possibilities for each of the n different balls.
3. Recall that there are 52 cards in a deck, and there are 13 cards in each of four suits. Further, the order of cards in a poker hand does not matter.
   (a) How many flushes (all cards of the same suit) are there in a 5 card poker hand?
   Answer: $4\binom{13}{5}$. We have 4 suits, and $\binom{13}{5}$ flushes in each.
   (b) Suppose you are playing Texas Hold'em, and you have two hearts, and five more cards from the remaining 50 will be flipped over. How many ways can at least 3 more hearts be present in the 5 new cards flipped over? (In other words, how many ways can the 7 total cards contain a flush?)
   Answer: $\binom{39}{2} \times \binom{11}{3} + \binom{39}{1} \times \binom{11}{4} + \binom{11}{5}$
4. How many integer solutions are there for $x_1 + \cdots + x_k = n$, where $x_i \geq -1$ for all i?
   Answer: $\binom{n+2k-1}{k-1} = \binom{n+2k-1}{n+k}$. We cannot use stars and bars directly here, since we'd like each of the terms to be nonnegative. As such, suppose we take $x_i' = x_i + 1$. Here, $x_i' \geq 0$ for all i, to make $x_1' + \cdots + x_k' = n + k$. Using stars and bars now, we have $n + k$ stars and $k - 1$ bars, giving $\binom{n+2k-1}{k-1} = \binom{n+2k-1}{n+k}$ possibilities.

9. Grids

Consider the following grid. A walk on this grid is only allowed to move right or up.



1. How many ways can you walk from A to B? (Note that it has width 7 and height 5.)
   Answer: $\binom{12}{5} = \binom{12}{7}$. There are exactly 12 steps that we must take, and we can choose 5 of those steps to go up (equivalently, we can choose 7 steps to go right).
2. How many ways can you walk from A to B while passing through P?
   Answer: $\binom{5}{2} \times \binom{7}{3}$. The paths that pass through P can be broken up into two parts: the segment from A to P and the segment from P to B. There are $\binom{5}{2}$ different paths from A to P (5 steps, 2 of which go up), and $\binom{7}{3}$ different paths from P to B (7 steps, 3 of which go up). Multiplying these two parts gives the final answer.

3. (8 points) Show that $\binom{12}{7} = \sum_{i=0}^{5} \binom{3+i}{i}\binom{8-i}{5-i}$ using a combinatorial proof. (Proofs based on calculations and algebra alone will not receive any points.)

   Answer: $\binom{12}{7}$ gives the number of paths that go from A to B. Looking at the RHS, consider all the possible edges e that go from column 3 to 4—that is, we can move edge e within its column in the grid. Any path from A to B must pass through some edge between these columns, so it suffices to compute the number of paths from A to B passing through all possible edges in this column.

   If the edge is between $(3,i)$ and $(4,i)$, then it takes 3 horizontal steps and i vertical steps to get to its endpoint and one must choose i places for the vertical step. To get from $(4,i)$ to the end it takes 3 horizontal steps and $5-i$ vertical steps. Thus one sums over i of the terms $\binom{3+i}{i}\binom{8-i}{5-i}$

10. Counting Polynomials.

Recall that a polynomial is defined as $f(x) = \sum_{k=0}^{n} a_k x^k$, where a_k's are a finite list of coefficients. We will extend this definition to *infinite polynomials*, which take the form $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where a_k's are an infinite list of coefficients.

Recall that two polynomials f and g are *equivalent* if for all x, $f(x) = g(x)$.

Classify the following sets as either finite, countably infinite, or uncountably infinite.

1. The set of all non-equivalent finite polynomials with integer coefficients.

   Answer: Countably infinite; each can be specified as a finite list of integers $(a_0, \ldots, a_n)$.

2. The set of all non-equivalent finite polynomials with real coefficients.

   Answer: Uncountably infinite; just the set of degree 0 polynomials is uncountably infinite.

3. For a fixed prime p, the set of all non-equivalent finite polynomials over modulo p.

   Answer: Finite; all polynomials with degree greater than p are equivalent to another polynomial with degree less than p, and each polynomial can be specified by a finite list of integers $(a_0, \ldots, a_n)$ modulo p.

4. The set of all non-equivalent *infinite* polynomials with integer coefficients.

   Answer: Uncountably infinite; by diagonalization. Even the set of infinite polynomials with coefficients in $\{0,1\}$ is uncountably infinite, since we have a bijection to the set of infinite bitstrings.

5. The set of all non-equivalent *infinite* polynomials with real coefficients.

   Answer: Uncountably infinite; finite polynomials with real coefficients are uncountably infinite as well.

6. For a fixed prime p, the set of all non-equivalent *infinite* polynomials over modulo p.

   Answer: Finite; same reasoning as with finite polynomials—terms with degree larger than p are all equivalent to terms with degree less than p.

11. Computability. (2 points each)

Given a program P and an input x, you want to determine whether $P(x)$ ever calls the subroutine Q. We'll show that this problem is uncomputable.

Let `TestCalls(P, Q, x)` be a program that determines if $P(x)$ ever calls the subroutine Q.

Fill in the blanks below to implement `TestHalt` using `TestCalls`, utilizing only variables defined in the template. You may assume that `dummy` is a unique subroutine that is not present in `F`.

```
def TestHalt(F, a):
    def dummy():
        pass  # do nothing

    def inner(b):
        _______(1)_______
        _______(2)_______

    return TestCalls(______(3)______, ______(4)______, a)
```

(1) **Answer:** `F(a)` or `F(b)`, since the input to `inner` is `b` from the call to `TestCalls`.

(2) **Answer:** `dummy()`

(3) **Answer:** `inner`

(4) **Answer:** `dummy`

12. Probability Theory.

1. Let A, B, and C be events, where $\mathbb{P}[A] = \frac{1}{2}$, $\mathbb{P}[B] = \frac{1}{3}$, and $\mathbb{P}[C] = \frac{1}{6}$. Suppose further that $\mathbb{P}[A \cup B] = \frac{2}{3}$ and $\mathbb{P}[A \mid C] = \frac{2}{3}$.
   (a) What is $\mathbb{P}[A \cap B]$?
   Answer: $\frac{1}{6}$. $\mathbb{P}[A \cap B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cup B] = \frac{1}{2} + \frac{1}{3} - \frac{2}{3} = \frac{1}{6}$ by inclusion/exclusion.
   (b) What is $\mathbb{P}[B \mid A]$?
   Answer: $\frac{1}{3}$. $\mathbb{P}[B \mid A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{1/6}{1/2} = \frac{1}{3}$.
   (c) What is $\mathbb{P}[C \cap A]$?
   Answer: $\frac{1}{9}$. $\mathbb{P}[C \cap A] = \mathbb{P}[A \mid C]\mathbb{P}[C] = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$.
   (d) What is $\mathbb{P}[C \mid A]$?
   Answer: $\frac{2}{9}$. $\mathbb{P}[C \mid A] = \frac{\mathbb{P}[C \cap A]}{\mathbb{P}[A]} = \frac{1/9}{1/2} = \frac{2}{9}$.

2. Suppose I_D, and I_E are indicator random variables for the events D and E, where $\mathbb{P}[D] = 1/3$, $\mathbb{P}[E] = 1/2$, and $\mathbb{P}[D \cap E] = 1/5$.
   (a) What is $\text{cov}(I_D, I_E)$?
   Answer: $\frac{1}{30}$. $\text{cov}(I_D, I_E) = \mathbb{E}[I_D I_E] - \mathbb{E}[I_D]E[I_E] = \mathbb{P}[D \cap E] - \mathbb{P}[D]\mathbb{P}[E] = \frac{1}{5} - \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{30}$.
   (b) Recall $\text{LLSE}[I_D \mid I_E] = a(I_E - \mathbb{E}[I_E]) + b$ for constants a and b.
      i. Is the constant a positive or negative or zero?
      Answer: Positive, since the covariance is positive. When $I_E = 1$, than D is more likely than otherwise.
      ii. What is the value of a?
      Answer: $\frac{2}{15}$. The coefficient is

$$\frac{\text{cov}(I_D, I_E)}{\text{Var}(I_E)} = \frac{1/30}{1/4} = \frac{2}{15}$$

      iii. What is the value of b?
      Answer: $\frac{1}{3}$. It should be the $\mathbb{E}[I_D]$, as if I_E is at its expectation, one should just predict $\mathbb{E}[I_D]$.

3. Suppose X and Y are independent random variables, where $\mathbb{E}[X] = 2$, $\mathbb{E}[Y] = 3$, $\text{Var}(X) = 1$, and $\text{Var}(Y) = 2$. (The following answers should be numerical values.)
   (a) What is $\mathbb{E}[XY]$?
   Answer: 6. $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 6$, since X and Y are independent.
   (b) What is $\mathbb{E}[X^2]$?
   Answer: 5. $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 5$.
   (c) What is $\text{Var}(XY)$?
   Answer: 19.
$$\mathbb{E}[(XY)^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] = (\text{Var}[X] + \mathbb{E}[X]^2)(\text{Var}[Y] + \mathbb{E}[Y]^2) = (1+4)(2+9) = 55.$$
   and
$$\text{Var}(XY) = \mathbb{E}[(XY)^2] - \mathbb{E}[X]^2\mathbb{E}[Y]^2 = 55 - 4 \times 9 = 19$$

4. Let X be a random variable with PDF $f_X(x) = x$ for $x \in [0, c]$ and $f_X(x) = 0$ otherwise.
   (a) What is c?
   Answer: $\sqrt{2}$. The $\int_0^c x\,\mathrm{d}x = \frac{x^2}{2}\Big|_0^c = c^2/2 = 1$, and thus $c = \sqrt{2}$.
   (b) What is the CDF $F_X(x)$ of X, for $x \in [0, c]$?
   Answer: $\frac{x^2}{2}$. We used this above.

13. Distributions

1. Consider two independent binomial random variables $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(n, q)$.
   (a) What is $\text{cov}(X, Y)$?
   Answer: 0. The variables are independent.
   (b) What is $\text{Var}(X + Y)$?
   Answer: $n((p)(1-p) + (q)(1-q))$. Variance of the binomials add since independent.
2. For a geometric random variable $X \sim \text{Geom}(p)$, what is $\mathbb{E}[X - x \mid X \geq x]$, for $x \geq 0$?
   Answer: $1/p$. Geometric distribution is memoryless.
3. For a Poisson random variable $X \sim \text{Poisson}(\lambda)$, what is $\mathbb{P}[X > 0]$?
   Answer: $1 - e^{-\lambda}$. Since $\mathbb{P}[X > 0] = 1 - \mathbb{P}[X = 0]$ and $\mathbb{P}[X = 0] = e^{-\lambda}$.
4. For independent Poisson random variables $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\lambda')$, what is $\mathbb{P}[X + Y > 0]$?
   Answer: $1 - e^{-\lambda-\lambda'}$. Since $\mathbb{P}[X + Y > 0] = 1 - \mathbb{P}[X + Y = 0]$, since $X + Y \sim P(\lambda + \lambda')$ we have the result.
5. For an exponential random variable $X \sim \text{Exp}(\lambda)$, what is its PDF $f_X(x)$ for $x \geq 0$?
   Answer: $f_X(x) = \lambda e^{-\lambda x}$. Its the right form ($\propto e^{-\lambda}$) and $\int_0^\infty \lambda e^{-\lambda x}\,\mathrm{d}x = -e^{-\lambda x}\big|_0^\infty = 1$.
6. For independent exponential random variables $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\lambda)$, what is the PDF $f_Z(z)$ for $Z = \min(X, Y)$? (*Hint*: what is the probability that both X and Y are greater than z?)
   Answer: $2\lambda e^{-2\lambda z}$. $\mathbb{P}[\min(X, Y) \geq z] = (e^{-\lambda z})^2$ and thus the CDF is $1 - e^{-2\lambda z}$.
7. Consider a dart hitting a dartboard of radius r, uniformly over its area. Let X be the distance from the dart to the center of the dartboard. What is the PDF $f_X(x)$ of X?
   Answer: $\frac{2x}{r^2}$. One can see the CDF is $\frac{\pi x^2}{\pi r^2}$ as the area of the part of the dartboard within x over the total. And take the derivative.

14. Uniform Distribution

Consider continuous uniform random variables X, Y that are independently distributed as $U[0, r]$.

1. Compute the CDF $F_Z(z)$ of $Z = \max(X, Y)$.
   Answer: $F_Z(z) = \begin{cases} 0 & \text{if } z < 0, \\ (z/r)^2 & \text{if } 0 \le z \le r, \\ 1 & \text{else.} \end{cases}$
2. Compute the PDF $f_Z(z)$ of $Z = \max(X, Y)$.
   Answer: $f_Z(z) = \begin{cases} 2z/r^2 & \text{if } 0 \le z \le r, \\ 0 & \text{else.} \end{cases}$
3. Compute $\mathbb{E}[\max(X, Y)]$.
   Answer: $\int_0^r \frac{2z^2}{r^2}\, dz = \frac{1}{r^2}\frac{2}{3}z^3 \Big|_0^r = \frac{2}{3}r$
4. Compute $\mathbb{E}[\min(X, Y)]$. (*Hint*: you do not have to compute any integrals or find the distribution of $\min(X, Y)$.)
   Answer: $\frac{1}{3}r$. By symmetry, this is $\mathbb{E}[r - \max(X, Y)]$. Alternately, since $X + Y = \min(X, Y) + \max(X, Y)$, then $\mathbb{E}[\min(X, Y)] = \mathbb{E}[X + Y] - \mathbb{E}[\max(X, Y)]$, thus $\mathbb{E}[\min(X, Y)] = \frac{1}{3}r$
5. Compute $\mathbb{E}[|X - Y|]$. (*Hint*: you do not have to compute any integrals; try relying on linearity of expectation.)
   Answer: Because $|X - Y| = \max(X, Y) - \min(X, Y)$, then $\mathbb{E}|X - Y| = \frac{2}{3}r - \frac{1}{3}r = \frac{1}{3}r$

15. Close to Home

Suppose we have balls numbered 1 through 100 in bins numbered 1 through 100. Suppose we then shuffle the balls by uniformly permuting them in the 100 bins, such that each bin has exactly one ball.

After shuffling, we say that a ball is "close to home" if it lands within 1 step of its original position. For example:

- Ball 4 is close to home if and only if it lands in bin 3, 4, or 5.
- Ball 100 is close to home if and only if it lands in bin 99 or 100.

1. Let X be the number of balls that are "close to home" after shuffling. What is $\mathbb{E}[X]$?
   Answer: Let I_j be the indicator variable for whether ball j is close to home. For balls 2 through 99, there are 3 bins (out of 100) that count as close to home, so $\mathbb{E}[I_j] = \frac{3}{100}$. For ball 1 and ball 100, there are only 2 bins that count as close to home, so $\mathbb{E}[I_1] = \mathbb{E}[I_{100}] = \frac{2}{100}$. Hence,

$$\mathbb{E}[X] = \sum_{j=1}^{100} \mathbb{E}[I_j] = \left(\frac{3}{100} \times 98\right) + \left(\frac{2}{100} \times 2\right) = \frac{294}{100} + \frac{4}{100} = \frac{298}{100} = 2.98.$$

2. Let I_i be an indicator for whether the ith ball is close to home after shuffling.
   (a) What is $\mathbb{E}[I_2 I_5]$?
   Answer: $\frac{9}{100 \times 99}$. The expectation is the probability that both balls are close to home. Ball 2 is close to home if it is in bins $[1, 2, 3]$ and Ball 5 is close to home if it is in bins $[4, 5, 6]$. There are 9 such cases, thus there are 9 possibilities out of 99×100 ways of assigning the two balls to the bins.

(b) What is $\mathbb{E}[I_2 I_4]$?
Answer: $\frac{8}{100\times 99}$. The expectation is the probability that both balls are close to home. Ball 2 is close to home if it is in bins $[1,2,3]$ and 4 is close to home if it is in bins $[3,4,5]$. There are 9 such cases, but both can't be in 3, thus there are 8 possibilities out of 99×100 ways of assigning the two balls to the bins.

(c) Are I_i and I_j independent for any $i,j\in\{1,\dots,100\}$?
Answer: No. $\mathbb{E}[I_i]\mathbb{E}[I_j]=\frac{9}{100^2}$ for non-corner i,j which is different than either of the expressions above, and the other cases will work out similarly. Intuitively, two balls that share a bin will be negatively correlated, and two balls that do not share a bin will be positively correlated - thus there cannot be any two balls with no correlation, and therefore they cannot be independent.

16. Which deck?

A dealer picks one of two decks with equal probability. The loaded deck has 39 red cards and 13 black cards, the fair deck has 26 red cards and 26 black cards.

1. If the dealer deals a red card, what is the probability that the deck is fair?
Answer: $\mathbb{P}[F\mid R]=\frac{\mathbb{P}[F\cap R]}{\mathbb{P}[R]}=\frac{0.5\times 26/52}{0.5\times 26/52+0.5\times 39/52}=\frac{26}{65}=\frac{2}{5}$.
2. Let p be the answer to the previous part. Still assuming that the dealer deals a red card, what is the probability that the next card will be red? (No need to simplify your answer. You may express your answer in terms of p.)
Answer: $3/5\times 38/51+2/5\times 25/51$. Or, in terms of p, $(1-p)\times 38/51+p\times 25/51$.

17. Union Bound, Markov, and Chebyshev

1. (4 points) Let $X=\sum_{j=1}^{n} I_j$, where each I_j is some indicator variable. Let p be the probability that at least one indicator is true. Prove that $p\le\mathbb{E}[X]$.
Answer: Using the union bound, $p=\mathbb{P}[X\ge 1]=\mathbb{P}[X_1=1\cup X_2=1\cup\cdots\cup X_n=1]\le\mathbb{P}[X_1=1]+\mathbb{P}[X_1=1]+\cdots+\mathbb{P}[X_n=1]=\mathbb{E}[X_1]+\mathbb{E}[X_2]+\cdots+\mathbb{E}[X_n]=\mathbb{E}[X]$.
2. (4 points) Let W be the number of successes in n independent Bernoulli trials, each with success probability p. Prove that $\mathbb{P}[W\ge k]\le\frac{np}{k}$.
Answer: $\mathbb{E}[W]=np$ by the fact that a Bernoulli random variable has $E[B]=p$ and linearity of expectation. Then we can apply Markov: $\mathbb{P}[W\ge k]\le\frac{\mathbb{E}[W]}{k}=\frac{np}{k}$.
3. (5 points) Suppose $Y\sim\text{Poisson}(\lambda)$. Prove that $\mathbb{P}[Y\ge 10\lambda]\le\frac{1}{81\lambda}$.
Answer:

$$\mathbb{P}[Y\ge 10\lambda]=\mathbb{P}[Y-\lambda\ge 9\lambda]\le\mathbb{P}[|Y-\lambda|\ge 9\lambda]=\mathbb{P}[|Y-\mathbb{E}[Y]|\ge 9\lambda]$$

This means we can use Chebyshev's inequality to say that:

$$\mathbb{P}[Y\ge 10\lambda]\le\frac{\text{Var}(Y)}{81\lambda^2}=\frac{1}{81\lambda}$$

18. Conditional Probability. (2 + 7 points)

Consider this statement: For any events A, B, and C, we have $\mathbb{P}[C\mid A\cup B]=\mathbb{P}[C\mid A]+\mathbb{P}[C\mid B]-\mathbb{P}[C\mid A\cap B]$.

Is this true or false? If true, give a brief sketch of why this is true. If it is false, give a counterexample. In your counterexample, give the calculations for these four quantities.

Answer: This is false. While this superficially resembles the principle of inclusion-exclusion, the given formula is incorrect.

Consider two independent, fair coin tosses. Let the sample space be $\{HH, HT, TH, TT\}$, each outcome having probability $1/4$.

Define the events:

$$A = \{\text{first toss is Heads}\} = \{HH, HT\}$$

$$B = \{\text{second toss is Heads}\} = \{HH, TH\}$$

$$C = \{\text{both tosses are Heads}\} = \{HH\}$$

Now compute the required conditional probabilities:

1. $\mathbb{P}[C \mid A \cup B] = 1/3$: The only possible outcomes from our given are $\{HT, HH, TH\}$, and event C is exactly one out of these three equally likely outcomes.
2. $\mathbb{P}[C \mid A] = 1/2$. If one coin is heads, there's a 50% chance the next one is also heads.
3. $\mathbb{P}[C \mid B] = 1/2$. Same.
4. $\mathbb{P}[C \mid A \cap B] = 1$.

Now check the given formula:

$$\mathbb{P}[C \mid A] + \mathbb{P}[C \mid B] - \mathbb{P}[C \mid A \cap B] = \frac{1}{2} + \frac{1}{2} - 1 = 1 - 1 = 0.$$

However,

$$\mathbb{P}[C \mid A \cup B] = \frac{1}{3}.$$

Since $1/3 \neq 0$, the proposed identity does not hold in this case.

19. Confidence

Consider independent $X_1, \ldots, X_n$, where each $X_i \sim \text{Poisson}(\lambda)$. (Recall $\mathbb{E}[X_i] = \lambda$ and $\text{Var}(X_i) = \lambda$.) Your answers to the following parts should all be in terms of λ.

1. Suppose $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. What is $\mathbb{E}[A_n]$?
   Answer: λ
2. Use Chebyshev's inequality to find the value of n that ensures that

   $$\mathbb{P}\left[|A_n - \mathbb{E}[A_n]| \geq 0.1\mathbb{E}[A_n]\right] \leq 0.05.$$

   Answer: $2000/\lambda$. Chebyshev implies

   $$\mathbb{P}[|A_n - \mathbb{E}[A_n]| \geq .1\mathbb{E}[A_n]] \leq \frac{\text{Var}(A_n)}{(.1\mathbb{E}[A_n])^2}.$$

   And $\text{Var}(A_n) = \frac{1}{n^2}\sum_i \text{Var}(X_i) = \frac{\text{Var}(X_1)}{n} = \frac{\lambda}{n}$. Also $\mathbb{E}[A_n] = \lambda$. Thus, we have

   $$\mathbb{P}[|A_n - \mathbb{E}[A_n]| \geq 0.1\mathbb{E}[A_n]] \leq \frac{\lambda/n}{(0.1\lambda^2)} \leq \frac{100}{n\lambda} \leq 0.05$$

   Solving yields $n \geq 2000/\lambda$.

3. Suppose A_n is normally distributed as $N(\mu, \sigma^2)$ for $\mu = \mathbb{E}[A_n]$ and $\sigma^2 = \mathrm{Var}(A_n)$. Further, recall that for a standard normal Z, $\mathbb{P}[|Z| \geq 2] \leq 0.05$. Find n that ensures that

$$\mathbb{P}\big[|A_n - \mathbb{E}[A_n]| \geq 0.1\mathbb{E}[A_n]\big] \leq 0.05.$$

Answer: Here we have $\sigma^2 = \frac{\lambda}{n}$. And need

$$0.1\mathbb{E}[A_n] = 0.1\lambda \geq 2\sigma = 2\sqrt{\frac{\lambda}{n}}.$$

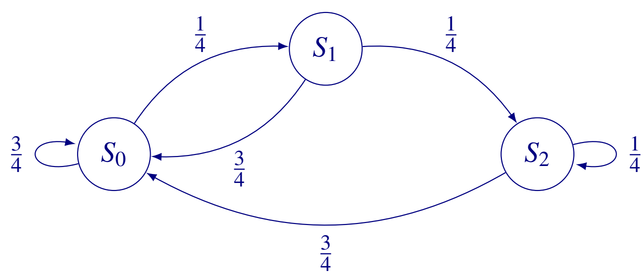
Note that the direction of the inequality is because we want $0.1\mathbb{E}[A_n]$ to be a z score of at least 2, not at most 2.
This yields $n \geq 400/\lambda$.

20. Markov Models

Suppose we repeatedly roll a four sided die and wish to track whether our last roll or last two rolls were ones. This process can be modeled by a Markov Chain with three states:

- State 0, where the previous roll was not a one.
- State 1, where the previous roll was a one, but the previous two rolls were not ones.
- State 2, where the previous two rolls were both ones.

1. (4 points) We wish to find the expected number of times we roll an even number before rolling two ones in a row. Write the first-step equations for this problem. (Do not solve the system of equations.)
Answer: For reference, here is the Markov chain:



Let $\beta(C)$ be the expected number of even numbers we see, given that we are currently at State C.

$$\beta(0) = \frac{1}{4}\beta(1) + \frac{1}{4}(1+\beta(0)) + \frac{1}{4}\beta(0) + \frac{1}{4}(1+\beta(0))$$
$$\beta(1) = \frac{1}{4}\beta(2) + \frac{1}{4}(1+\beta(0)) + \frac{1}{4}\beta(0) + \frac{1}{4}(1+\beta(0))$$
$$\beta(2) = 0$$

Solving (which was not asked for) will yield $\beta(0) = 10$.
Note that it is fine to leave out the third equation, and just write in the 0 in the second equation. These equations are a bit confusing at first glance, but they follow the form

$$\begin{aligned}\beta(\text{current state}) =& \mathbb{P}[\text{roll a 1}]\beta(\text{current state + 1, since we successfully rolled a 1}) \\ &+ \mathbb{P}[\text{roll a 2}](1+\beta(\text{state 0, since we failed to roll a 1}]) \\ &+ \mathbb{P}[\text{roll a 3}]\beta(\text{state 0, since we failed to roll a 1}] \\ &+ \mathbb{P}[\text{roll a 4}](1+\beta(\text{state 0, since we failed to roll a 1}])\end{aligned}$$

And we only add a $+1$ when we roll a 2 or 4.

This may look different than the standard hitting time equations we use. Alternatively, we can alter our formulation to look more like the standard equations. In this case, let $\beta(C)$ be the expected number of even numbers we see, given that we just transitioned into state C.

$$\beta(S) = \frac{1}{4}\beta(1) + \frac{1}{4}\beta(0) + \frac{1}{4}\beta(0) + \frac{1}{4}\beta(0)$$
$$\beta(0) = \frac{2}{3} + \frac{1}{4}\beta(1) + \frac{1}{4}\beta(0) + \frac{1}{4}\beta(0) + \frac{1}{4}\beta(0)$$
$$\beta(1) = \frac{1}{4}\beta(2) + \frac{1}{4}\beta(0) + \frac{1}{4}\beta(0) + \frac{1}{4}\beta(0)$$
$$\beta(2) = 0$$

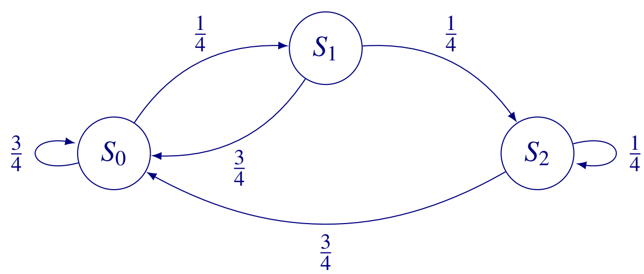
Only $\beta(0)$ adds a constant, and we add a $+2/3$ because $\beta(0)$ represents that we just rolled a number that was not a 1, which thus has a $2/3$ chance of being an even number. $\beta(1), \beta(2)$ don't add anything because we just rolled a 1. Note that we have to add an additional starting state to our system of equations because we don't start off at $\beta(0)$ anymore, since we don't start our chain having seen a roll. Solving this chain will yield $\beta(S) = 10$ as well.

2. Is there a unique stationary distribution $\pi = \begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}$?

   Answer: Yes, because the chain is irreducible.

3. (4 points) Either argue that there is no unique stationary distribution, or set up a system of equations for finding $\pi = \begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}$.

   Answer: Again, for reference, here is the Markov chain:



The balance equations are:

$$\pi(0) = \frac{3}{4}\pi(0) + \frac{3}{4}\pi(1) + \frac{3}{4}\pi(2)$$
$$\pi(1) = \frac{1}{4}\pi(0)$$
$$\pi(2) = \frac{1}{4}\pi(1) + \frac{1}{4}\pi(2)$$
$$1 = \pi(0) + \pi(1) + \pi(2)$$

Only 2 out of the first 3 equations are required, it is fine to leave out one of them.

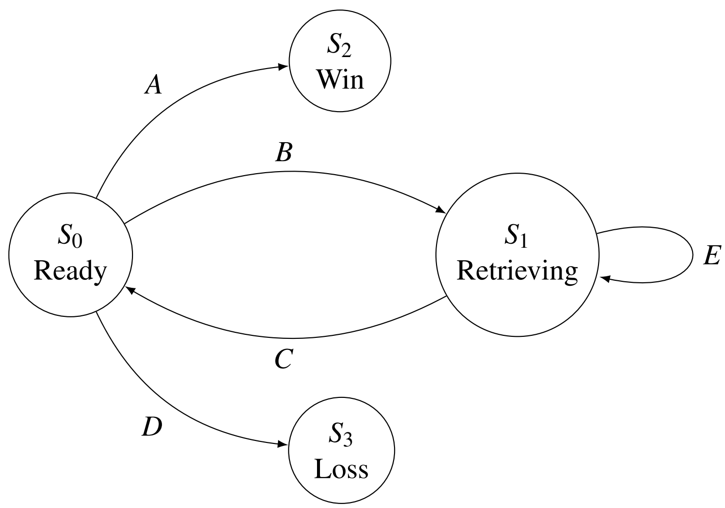
The solution to this system (not required for credit) is $\pi = \begin{bmatrix}\frac{3}{4} & \frac{3}{16} & \frac{1}{16}\end{bmatrix}$.

21. Collaborative Basketball.

Two friends are playing basketball, and they each have a separate basketball hoop. At each timestep, both attempt to shoot the basketball into their hoop, where each makes their shot with an independent probability of $\frac{2}{3}$.

However, if one friend misses, they spend that turn retrieving the basketball, and the other will take a shot that turn. Note that if the friend that shoots misses, they will take the next turn retrieving. If, at a given timestep, they both shoot and make their shots, then they win the game. However, if they both shoot and miss their shots, then they lose the game.

We can model this game as a Markov chain, shown below.

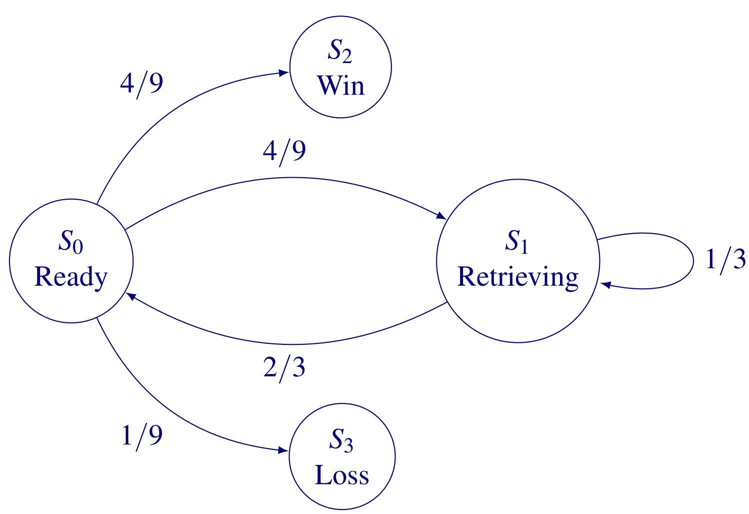


1. (1 point each) Fill in the transition probabilities A through E below.

$$A = \qquad B = \qquad C = \qquad D = \qquad E =$$

2. (4 points) What is the probability of winning?
3. (4 points) What is the expected number of timesteps it takes to terminate the game?

Answer:



1.

2. 4/5 by noticing that state 1 is irrelevant here, so only the relative weights of winning or losing matter, or by solving the α equations.
3. $\beta(S_0) = 1 + \frac{4}{9}\beta(S_1) + \frac{5}{9} \times 0$ and $\beta(S_1) = 1 + \frac{1}{3}\beta(S_1) + \frac{2}{3}\beta(S_0)$. Solving, we get $\beta(S_0) = 3$