1 Countability Intro

Note 11

A function $f : A \to B$ maps elements from set A to set B.

f is *injective* if it maps distinct elements to distinct elements, and *surjective* if every element in B is mapped to by some element in A. If f is both injective and surjective, it is *bijective*, and the sets A and B are said to have the same *cardinality* (size). The cardinality of a set is denoted by $|A|$.

f is bijective if and only if there exists an inverse function $f^{-1} : B \to A$ such that $f^{-1}(f(a)) = a$ for all $a \in A$ and $f(f^{-1}(b)) = b$ for all $b \in B$.

Countability: Formal notion of different kinds of infinities.

- *Countable*: able to enumerate in a list (possibly finite, possibly infinite)
- *Countably infinite*: able to enumerate in an infinite list; that is, there is a bijection with $\mathbb{N}$.

  To show that there is a bijection, the *Cantor–Bernstein theorem* says that it is sufficient to find two injections, $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$. Intuitively, this is because an injection $f : S \to \mathbb{N}$ means $|S| \le |\mathbb{N}|$, and an injection $g : \mathbb{N} \to S$ means $|\mathbb{N}| \le |S|$; together, we have $|\mathbb{N}| = |S|$.
- *Uncountably infinite*: unable to be listed out

  Use *Cantor diagonalization* to prove uncountability through contradiction; the classic example is the set of reals in $[0, 1]$:

$\mathbb{N}$	$[0,1]$
0	0 . ⑦ 3 2 0 5 0 ···
1	0 . 4 ① 4 2 1 3 ···
2	0 . 6 1 ⑧ 0 3 3 ···
3	0 . 1 8 2 ⑧ 1 8 ···
4	0 . 1 4 1 5 ⑨ 2 ···
5	0 . 5 7 7 2 1 ⑤ ···
⋮	⋮
?	0 . 8 2 9 9 1 6 ···

  If we change the digits along the diagonal, the new decimal created is different from every single element in the list in at least one place, so it's not in the list—this is a contradiction.

Sometimes it can be easier to prove countability/uncountability through bijections with other countable/uncountable sets respectively. Common countable sets include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, finite length bitstrings, etc. Common uncountable sets include $[0, 1]$, $\mathbb{R}$, infinite length bitstrings, etc.

(a) Your friend is confused about how Cantor diagonalization doesn't apply to the set of natural numbers. They argue that natural numbers can be thought of as an infinite length string of digits, by padding each number with an infinite number of zeroes to the left. If we then assume by contradiction that we can list out the set of natural numbers with the padded 0's, we can change the digits along a diagonal, to create a new natural number not in the list.

$$\begin{array}{c|cccccc}
0 & \cdots & 0 & 0 & 0 & 0 & \textcircled{1} \\
1 & \cdots & 0 & 1 & 2 & \textcircled{3} & 4 \\
2 & \cdots & 5 & 2 & \textcircled{8} & 2 & 3 \\
3 & \cdots & 9 & \textcircled{4} & 3 & 2 & 1 \\
\vdots & & & & \vdots & & \\
\hline
? & \cdots & 1 & 5 & 9 & 4 & 2
\end{array}$$

What is wrong with this argument?

Solution:

(a) The issue here is that the newly created number is not necessarily a natural number. This number has an infinite number of digits (we'll always be changing the padded zeroes into some nonzero digits), while natural numbers must have a finite number of digits.

This means that when you perform Cantor diagonalization to show a set S is uncountable, it is imperative that you ensure that the newly created element is always still an element of S; otherwise, we cannot say anything about the result of the diagonalization.

2 Countability: True or False

Note 11

(a) The set of all irrational numbers $\mathbb{R} \backslash \mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.

(c) The set of real solutions for the equation $x + y = 1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $(f \circ g)(x) = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) f and g are injective (one-to-one) $\implies f \circ g$ is injective (one-to-one).

(e) f is surjective (onto) $\implies f \circ g$ is surjective (onto).

Solution:

(a) **True.** Proof by contradiction. Suppose the set of irrationals is countable. From Lecture note 10 we know that the set $\mathbb{Q}$ is countable. Since union of two countable sets is countable, this would imply that the set $\mathbb{R}$ is countable. But again from Lecture note 10 we know that this is not true. Contradiction!

(b) **True.** Multiplying both sides of the modular equation by 7 (the multiplicative inverse of 3 with respect to 10) we get $x \equiv 4 \pmod{10}$. The set of all intergers that solve this is

$S = \{10k+4 : k \in \mathbb{Z}\}$ and it is clear that the mapping $k \in \mathbb{Z}$ to $10k+4 \in S$ is a bijection. Since the set $\mathbb{Z}$ is countably infinite, the set S is also countably infinite.

(c) **False.** Let $S \subset \mathbb{R} \times \mathbb{R}$ denote the set of all real solutions for the given equation. For any $x' \in \mathbb{R}$, the pair $(x', y') \in S$ if and only if $y' = 1 - x'$. Thus $S = \{(x, 1-x) : x \in \mathbb{R}\}$. Besides, the mapping x to $(x, 1-x)$ is a bijection from $\mathbb{R}$ to S. Since $\mathbb{R}$ is uncountable, we have that S is uncountable too.

(d) **True.** Recall that a function $h : A \to B$ is injective iff $a_1 \neq a_2 \implies h(a_1) \neq h(a_2)$ for all $a_1, a_2 \in A$. Let $x_1, x_2 \in X$ be arbitrary such that $x_1 \neq x_2$. Since g is injective, we have $g(x_1) \neq g(x_2)$. Now, since f is injective, we have $f(g(x_1)) \neq f(g(x_2))$. Hence $f \circ g$ is injective.

(e) **False.** Recall that a function $h : A \to B$ is surjective iff $\forall b \in B, \exists a \in A$ such that $h(a) = b$. Let $g : \{0,1\} \to \{0,1\}$ be given by $g(0) = g(1) = 0$. Let $f : \{0,1\} \to \{0,1\}$ be given by $f(0) = 0$ and $f(1) = 1$. Then $f \circ g : \{0,1\} \to \{0,1\}$ is given by $(f \circ g)(0) = (f \circ g)(1) = 0$. Here f is surjective but $f \circ g$ is not surjective.

3 Counting Cartesian Products

Note 11

For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

(c) Consider a countably infinite number of finite sets: $B_1, B_2, \ldots$ for which each set has at least 2 elements. Prove that $B_1 \times B_2 \times \cdots$ is uncountable.

Solution:

(a) As shown in lecture, $\mathbb{N} \times \mathbb{N}$ is countable by creating a zigzag map that enumerates through the pairs: $(0,0), (1,0), (0,1), (2,0), (1,1), \ldots$. Since A and B are both countable, there exists a bijection between each set and a subset of $\mathbb{N}$. Thus we know that $A \times B$ is countable because there is a bijection between a subset of $\mathbb{N} \times \mathbb{N}$ and $A \times B : f(i,j) = (A_i, B_j)$. We can enumerate the pairs (a,b) similarly.

(b) Proceed by induction.
Base Case: $n = 2$. We showed in part (a) that $A_1 \times A_2$ is countable since both A_1 and A_2 are countable.
Induction Hypothesis: Assume that for some $n \in \mathbb{N}$, $A_1 \times A_2 \times \cdots \times A_n$ is countable.
Induction Step: Consider $A_1 \times \cdots \times A_n \times A_{n+1}$. We know from our hypothesis that $A_1 \times \cdots \times A_n$ is countable, call it $C = A_1 \times \cdots \times A_n$. We proved in part (a) that since C is countable and A_{n+1} are countable, $C \times A_{n+1}$ is countable, which proves our claim.

(c) Let us assume that each B_i has size 2. If any of the sizes are greater than 2, that would only make the cartesian product larger. Notice that this is equivalent to the set of infinite length binary strings, which was proven to be uncountable in the notes.

Alternatively, we could provide a diagonalization argument: Assuming for the sake of contradiction that $B_1 \times B_2 \times \cdots$ is countable and its elements can be enumerated in a list:

$$\begin{gathered}(b_{1,1}, b_{2,1}, b_{3,1}, b_{4,1}, \dots) \\ (b_{1,2}, b_{2,2}, b_{3,2}, b_{4,2}, \dots) \\ (b_{1,3}, b_{2,3}, b_{3,3}, b_{4,3}, \dots) \\ (b_{1,4}, b_{2,4}, b_{3,4}, b_{4,4}, \dots) \\ \vdots\end{gathered}$$

where $b_{i,j}$ represents the item from set B_i that is included in the jth element of the Cartesian Product. Now consider the element $(\overline{b_{1,1}}, \overline{b_{2,2}}, \overline{b_{3,3}}, \overline{b_{4,4}}, \dots)$, where $\overline{b_{i,j}}$ represents any item from set B_i that differs from $b_{i,j}$ (i.e. any other element in the set). This is a valid element that should exist in the Cartesian Product $B_1, B_2, \dots$, yet it is not in the enumerated list. This is a contradiction, so $B_1 \times B_2 \times \cdots$ must be uncountable.

4 Counting Functions

Note 11 Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

Solution:

(a) Uncountable: Let us assume the contrary and proceed with a diagonalization argument. If there are countably many such function we can enumerate them as

	0	1	2	3	...
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	...
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now go along the diagonal and define f such that $f(x) > f_x(x)$ and $f(y) > f(x)$ if $y > x$, which is possible because at step k we only need to find a number $\in \mathbb{N}$ greater than all the $f_j(j)$ for $j \in \{0, \dots, k\}$; for example, we could define such a function using

$$f(x) = \begin{cases} f_0(0) + 1 & x = 0 \\ \max(f_x(x), f(x-1)) + 1 & x > 0 \end{cases}$$

This function differs from each f_i and therefore cannot be on the list, hence the list does not exhaust all non-decreasing functions. As a result, there must be uncountably many such functions.

Alternative Solution: Look at the subset $\mathscr{S}$ of strictly increasing functions. Any such f is uniquely identified by its image which is an infinite subset of $\mathbb{N}$. But the set of infinite subsets of $\mathbb{N}$ is uncountable. This is because the set of all subsets of $\mathbb{N}$ is uncountable, and the set of all finite subsets of $\mathbb{N}$ is countable. So $\mathscr{S}$ is uncountable and hence the set of all non-decreasing functions must be too.

Alternative Solution 2: We can inject the set of infinitely long binary strings into the set of non-decreasing functions as follows. For any infinitely long binary string b, let $f(n)$ be equal to the number of 1's appearing in the first n-digits of b. It is clear that the function f so defined is non-decreasing. Also, since the function f is uniquely defined by the infinitely long binary string, the mapping from binary strings to non-decreasing functions is injective. Since the set of infinite binary strings is uncountable, and we produced an injection from that set to the set of non-decreasing functions, that set must be uncountable as well.

(b) Countable: Let D_n be the subset of non-increasing functions for which $f(0) = n$. Any such function must stop decreasing at some point (because $\mathbb{N}$ has a smallest number), so there can only be finitely many (at most n) points $X_f = \{x_1, \ldots, x_k\}$ at which f decreases. Let y_i be the amount by which f decreases at x_i, then f is fully described by $\{(x_1, y_1), \ldots, (x_k, y_k), (-1,0), \ldots, (-1,0)\} \in \mathbb{N}^n = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ (n times), where we padded the k values associated with f with $n-k$ $(-1,0)$s. In Lecture note 11, we have seen that $\mathbb{N} \times \mathbb{N}$ is countable by the spiral method. Using it repeatedly, we get $\mathbb{N}^{(2^l)}$ is countable for all $l \in \mathbb{N}$. This gives us that $\mathbb{N}^n$ is countably for any finite n (because $\mathbb{N}^n \subset \mathbb{N}^{(2^l)}$ where l is such that $2^l \geq n$). Hence D_n is countable. Since each set D_n is countable we can enumerate it. Map an element of D_n to (n, j) where j is the label of that element produced by the enumeration of D_n. This produces an injective map from $\cup_{n \in \mathbb{N}} D_n$ to $\mathbb{N} \times \mathbb{N}$ and we know that $\mathbb{N} \times \mathbb{N}$ is countable from Lecture note 11 (via spiral method). Now the set of all non-increasing functions is $\cup_{i \in \mathbb{N}} D_n$, and thus countable.