GRAPHS - 2

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LECTURE PLAN

* RECAP

* PLANAR GRAPHS
  → EULER'S FORMULA
  → NON PLANARITY OF $K_5, K_{3,3}$

  → GRAPH COLORING
    → PLANAR GRAPHS
      → 6-COLOR THEOREM
      → 5-COLOR THEOREM

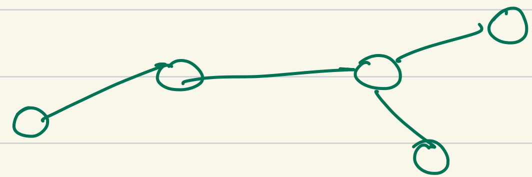
* HYPERCUBES:
  → RECURSIVE DEFINITION
  → CUTS & LARGE CUTS

REVIEW:

(Vertices, Edges)

Degree (vertex) = # of edges at vertex

TREES:

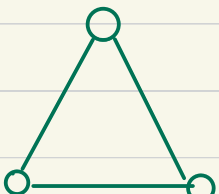


↳ Connected but no cycles

↳ Connected with $|V|-1$ edges V is number of vertices

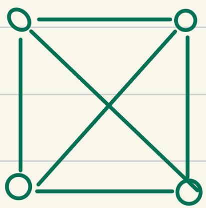
PLANAR GRAPHS : "Graphs that CAN BE DRAWN with NO EDGE CROSSINGS"

Which are PLANAR?



K_3

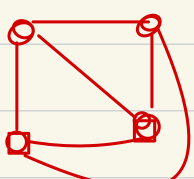
YES

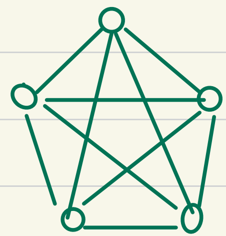


?

K_4

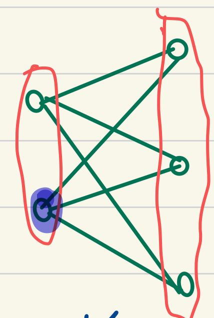
YES





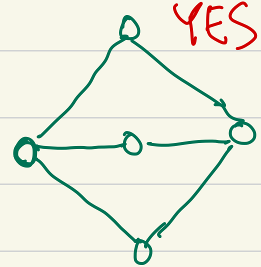
K_5

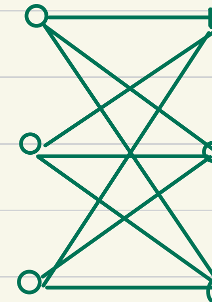
NO



$K_{2,3}$

YES



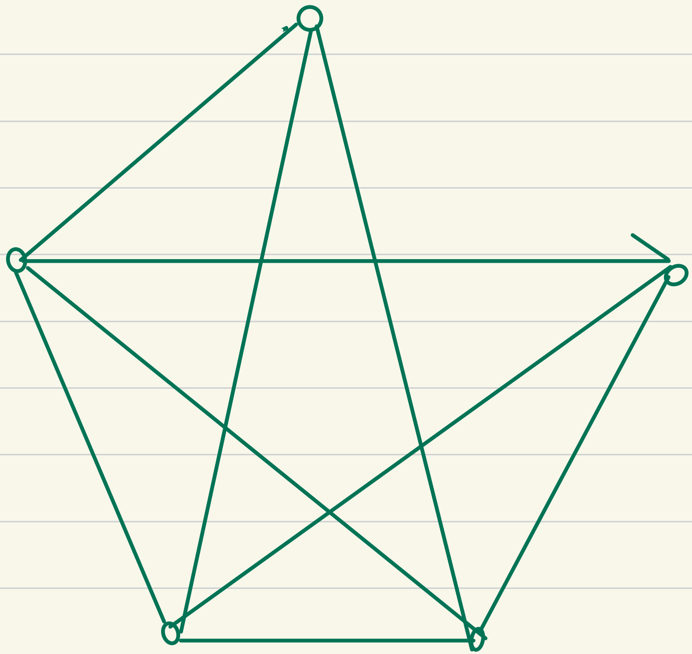


$K_{3,3}$

NO

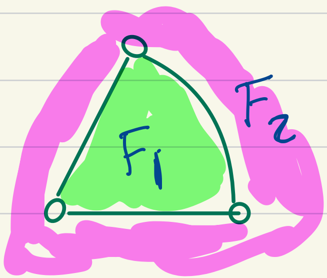
IS THIS GRAPH PLANAR?

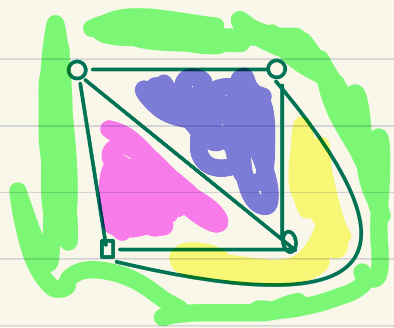
[EXERCISE]

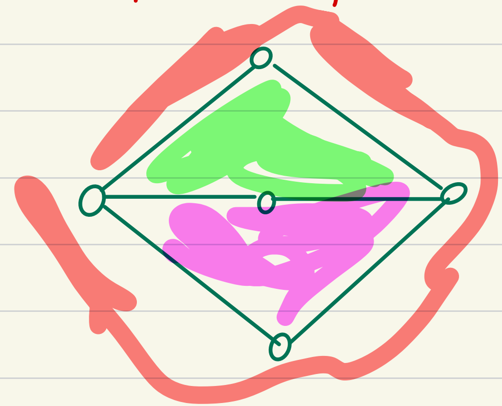


EULER'S FORMULA

DEFINITION: FACE = A connected region of the plane







FACES	2	4	3
VERTICES	3	4	5
EDGES	3	6	6

EULER'S FORMULA

In any connected planar graph

$$v + f = e + 2$$

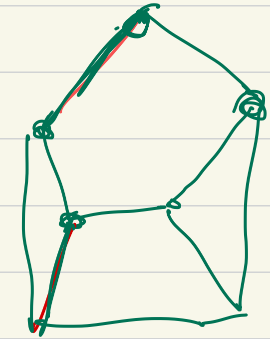
vertices faces edges

COROLLARY: IN ANY PLANAR GRAPH $e \leq 3v - 6$

edges vertices

PROOF:

Count face-edge adjacencies



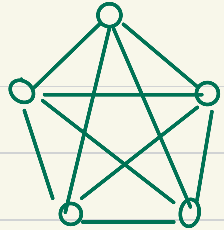
1) Every face has ≥ 3 adjacent edges

2) Every edge has $= 2$ adjacent faces

$$\Longrightarrow \quad 3f \leq 2e \quad \Rightarrow$$

$$\underbrace{e + 2 - v = f}_{\text{Euler's formula}} \leq 2e/3$$

$$e \leq 3v - 6$$

COROLLARY:  is not a planar graph.

K_5

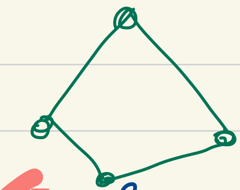
PROOF: $v = 5$ $\quad e = 10$

$e > 3v - 6$

$10 \quad\quad 9$

COROLLARY: In any planar graph $\boxed{\text{WITH NO TRIANGLES}}$ $e \leq 2v-4$

PROOF:



1) Every face is adjacent to at least 4 edges

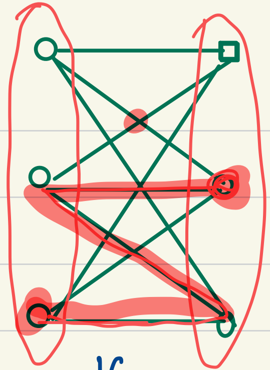
2) Every edge is adjacent to exactly 2 faces.

$$\Rightarrow 4f \leq 2 \cdot e$$

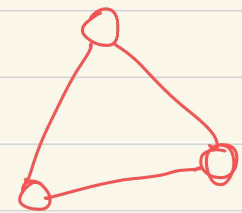
By Euler's formula $f = e+2-v$

$$e+2-v \leq e/2 \Rightarrow \boxed{e \leq 2v-4}$$

in a PLANAR GRAPH WITH NO $\triangle$les

COROLLARY:  is not a planar graph.

$K_{3,3}$



PROOF: → $K_{3,3}$ has no triangles in the graph.

$e = 9 \qquad v = 6$

$\underset{9}{e} \geq 2v - 4$ → violates Corollary

$\qquad 2 \cdot 6 - 4 = 8$

EULER'S FORMULA

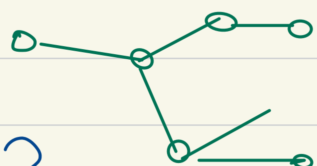
IN ANY CONNECTED PLANAR GRAPH $\quad v + f = e + 2$

PROOF: By induction, on number of edges e. [For every fixed v]

BASE CASE: Connected Graph has $\geq v-1$ edges. (TREES)

$e = v - 1$ (Graph is a TREE)

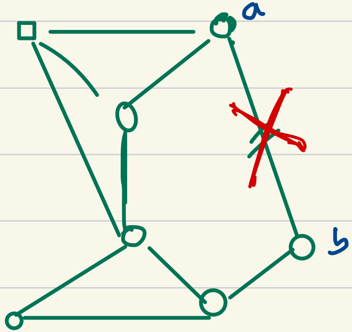
$f = 1$



$$\begin{array}{ccccccc} v & + & f & = & e & + & 2 \\ & & \shortparallel & & \shortparallel & & \\ v & + & 1 & = & (v-1) & + & 2 \end{array}$$

INDUCTION STEP: Euler's formula holds for all Graphs with $e = k$

Consider a graph G with $k+1$ edges. $\boxed{e > v-1}$



G has a cycle.

Pick an edge ab in a cycle.

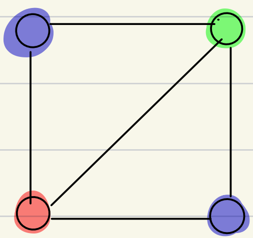
$$G' = G - (\text{edge } ab)$$

$$e' = e - 1$$
$$v' = v$$
$$f' = f - 1$$

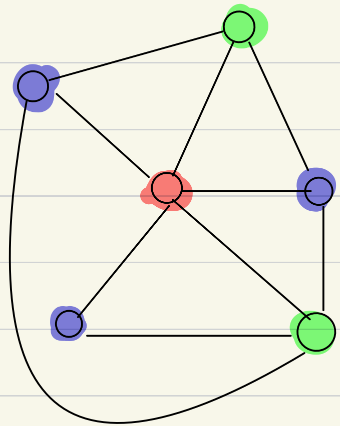
G' satisfies Euler's formula $\Rightarrow$ $v' + \underbrace{f'}_{+1} = \underbrace{e'}_{+1} + 2$

$$v + f = e + 2$$

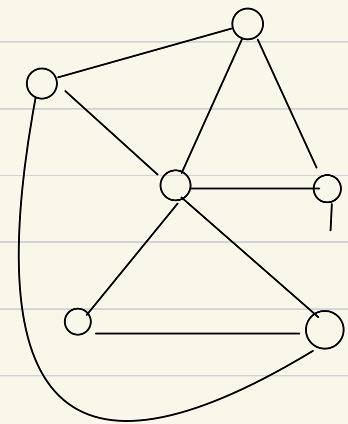
GRAPH COLORING: Assign colors to vertices such that for each edge, endpoints have different colors



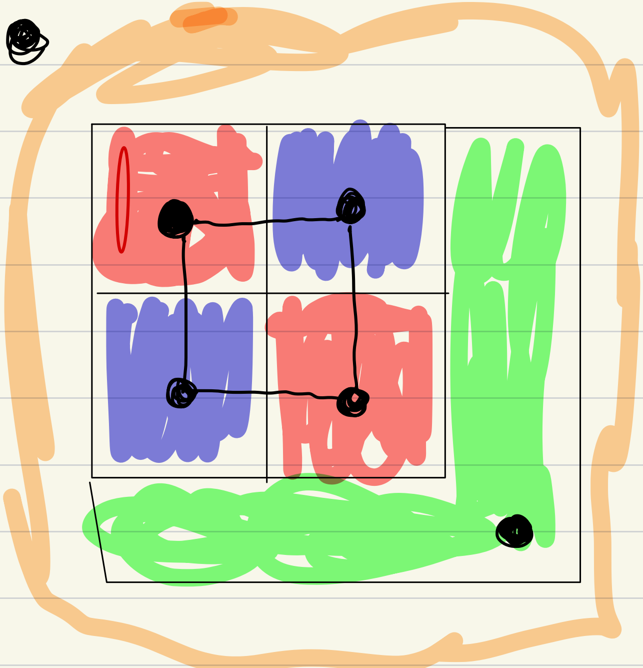
3 colors



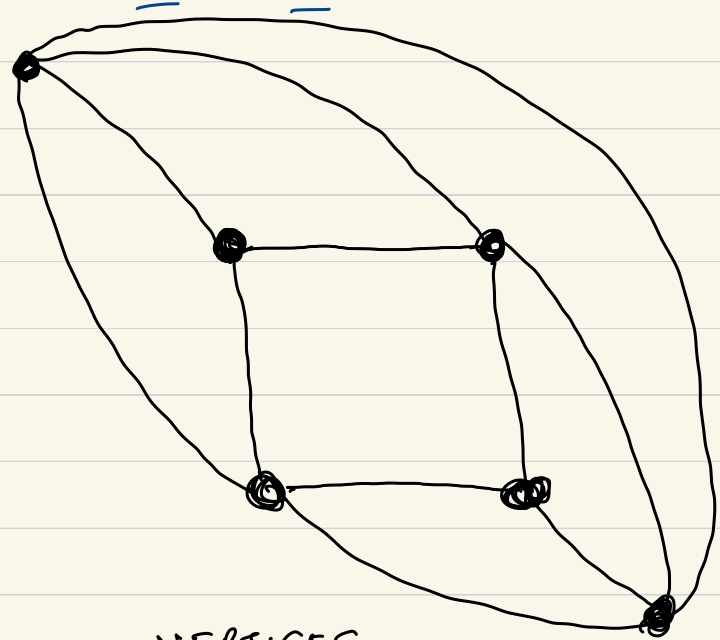
3 colors



COLORING MAPS $\equiv$ COLORING PLANAR GRAPHS



$\Longrightarrow$



REGIONS $\Longrightarrow$ VERTICES

ADJACENT REGIONS $\Longrightarrow$ EDGES.

Six Color Theorem: Every planar graph can be colored with 6 colors or less.

Induction on # of vertices v.

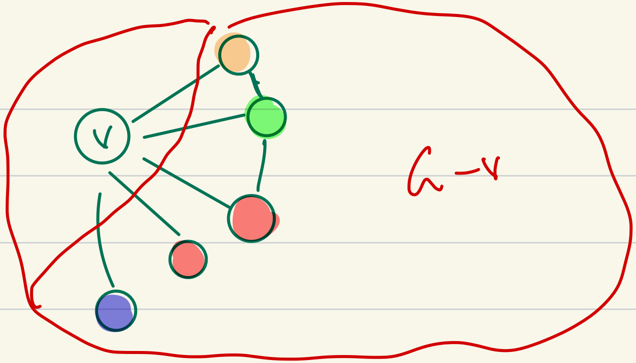
Proof: Recall, $e \leq 3v-6$ in a planar graph.

Total Degree of All Vertices $= 2e = 2(3v-6) = 6v-12$

$\Rightarrow$ Average degree $= \left(\frac{6v-12}{v}\right) < 6$

$\Rightarrow \exists$ Some vertex v degree$(v) \leq 5$.

Color $G-v$ inductively with 6 colors



v has 5 neighbors

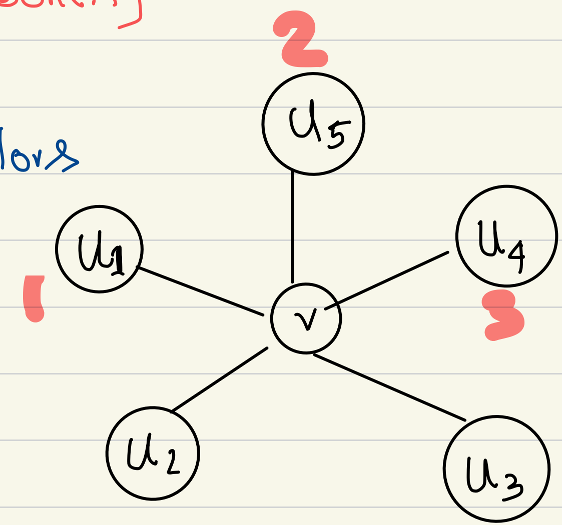
⟹ Out of 6 colors available one of colors is NOT used by any neighbor

⟹ COLOR v with UNUSED COLOR.

FIVE COLOR THEOREM: Every planar graph can be colored with 5 colors

PROOF: Again, there is a vertex v, degree$(v) \leq 5$.

[SIX COLOR THEOREM]



→ COLOR $G - v$ inductively. using 5 colors

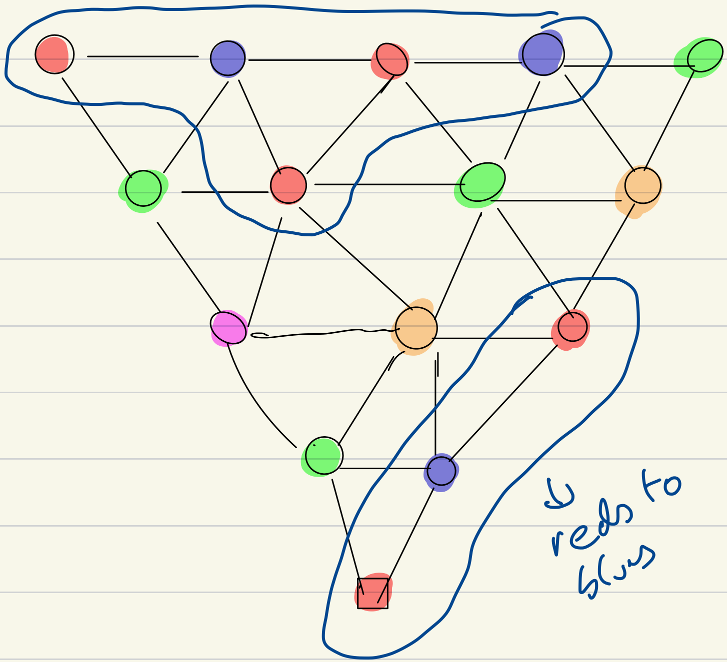
→ IF v has < 5 neighbors

$\exists$ an unused color for v

→ IF neighbors of v repeat colors

$\exists$ an unused color for v

COLOR SWITCH



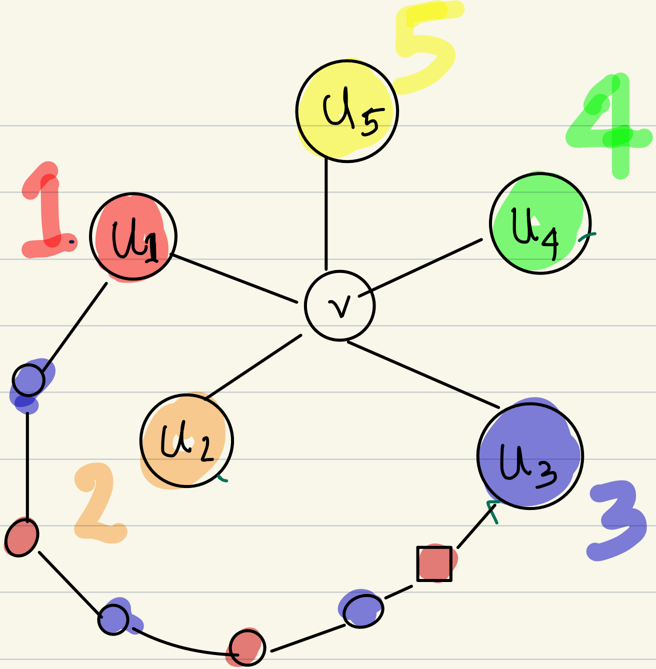
In a legal coloring,

1) Pick any two colors
red, blue

2) Consider connected components of two colors

3) Switch colors in any component.

New coloring is LEGAL.



Can we free up a COLOR ??

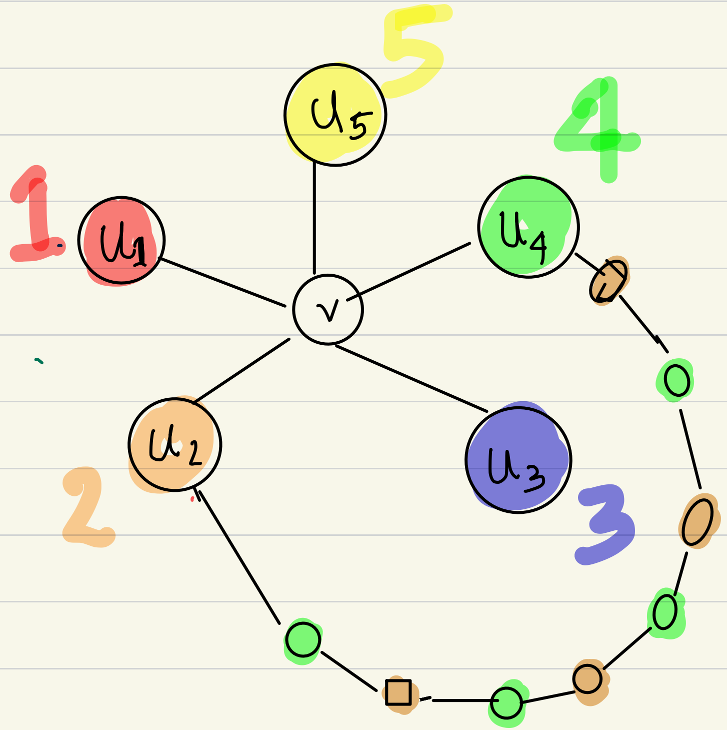
using (Color Switches)

→ Use a color switch between color 1 & 3 in component containing u_1

→ Frees up Color 1 UNLESS

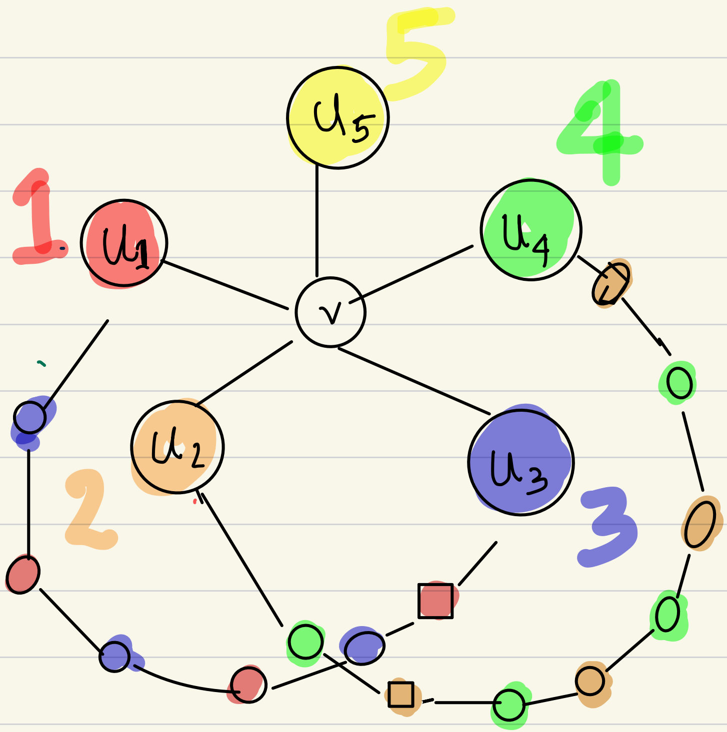
∃ a color 1-3 path from $u_1 \longrightarrow u_3$ in COLORS 1 & 3

Similarly $\exists$ a 2-4 path from u_2 to u_4 in colors 2 AND 4.

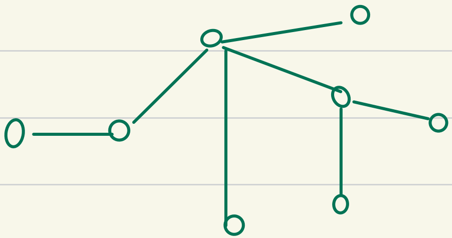


But then the Color 1-3 path & Color 2-4 path intersect!

A contradiction.



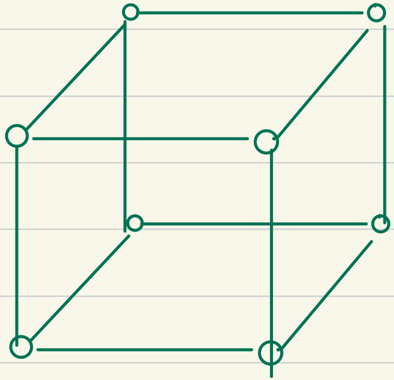
TREES



$e = V - 1$

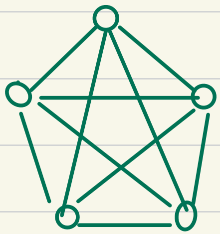
MINIMUM # of EDGES (while connected)

HYPERCUBES:



3-dim

COMPLETE GRAPH



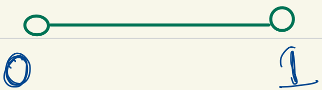
K_5

$e = \frac{V(V-1)}{2}$

Maximum # of edges

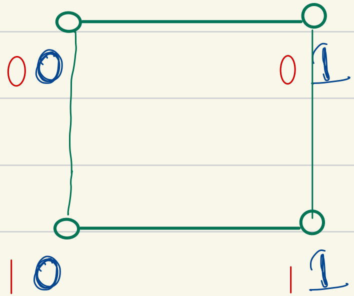
HYPERCUBES (INDUCTIVE DEFINITION)

$d = 1$



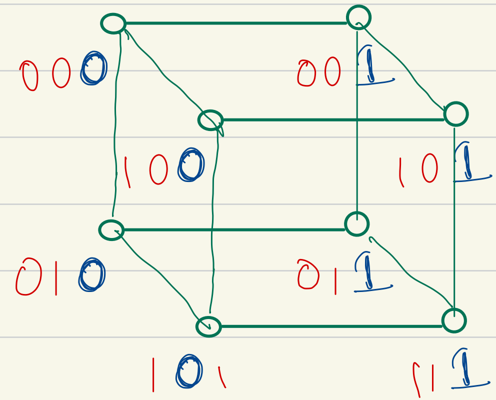
$v = 2$

$d = 2$



$v = 4$

$d = 3$



$v = 8$

$d = 4$ $v = 16$

$d - d$

d-dim hypercube

$v = 2^d$

$e = d \cdot 2^{d-1}$

HYPERCUBES:

DEFINITION:

d-dimensional hypercube graph H_d

VERTICES: ALL BINARY STRINGS ON d-bits $\{0,1\}^d$

EDGES: String x & String y are connected by an edge iff x & y differ in EXACTLY 1 bit.

$d = 3$

001
000
001
010
100
101
110
111

x is connected to all y
x & y differ in 1 bit.

degree (every vertex) = d in d-dimensional hypercube.

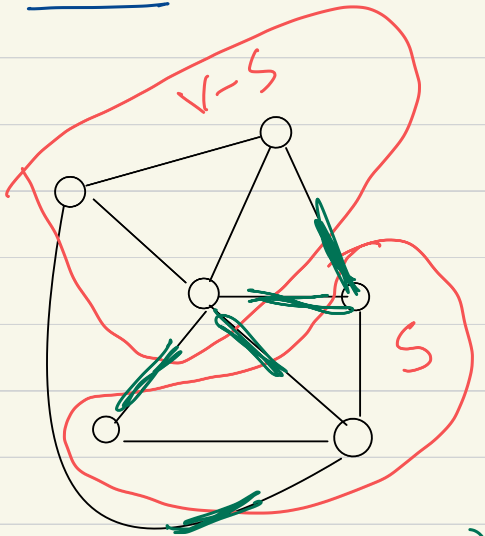
$2e$ = Total degree = $d \cdot 2^d$

$e = d \cdot 2^d / 2 = d \cdot 2^{d-1}$

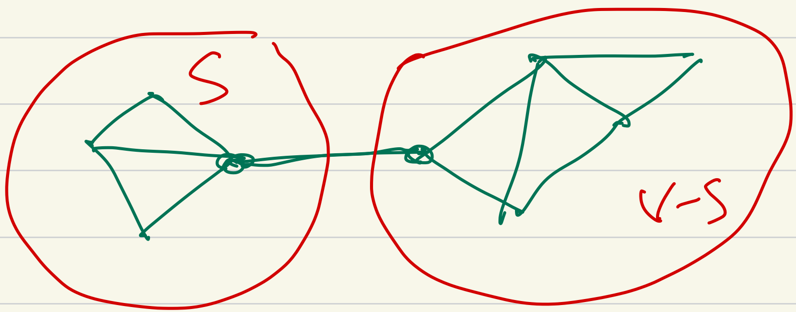
Cuts:

CUT = A partition (S, V-S) of vertices of graph.

Size of Cut (S, V \ S) = # of edges crossing from S to V-S



Size (S, V-S) = 5



Graph is very well-connected $\approx$ No cuts of small size

(PROPERTY)

THEOREM: In hypercube, SIZE OF ANY CUT $(S, V-S) \geq |S|$

where $|S| \leq |V-S|$