Announcements

- Load Balancing
  - Rohit’s session (MTuWTh 4-5pm) has ~40 students.

- Charlotte’s (MTuWTh 11am-12pm) and

  Lance’s sessions (MTuWTh 11am-12pm) have ~10 students.

  Anand’s session (MTuWTh 10am-11am) has ~15 students.

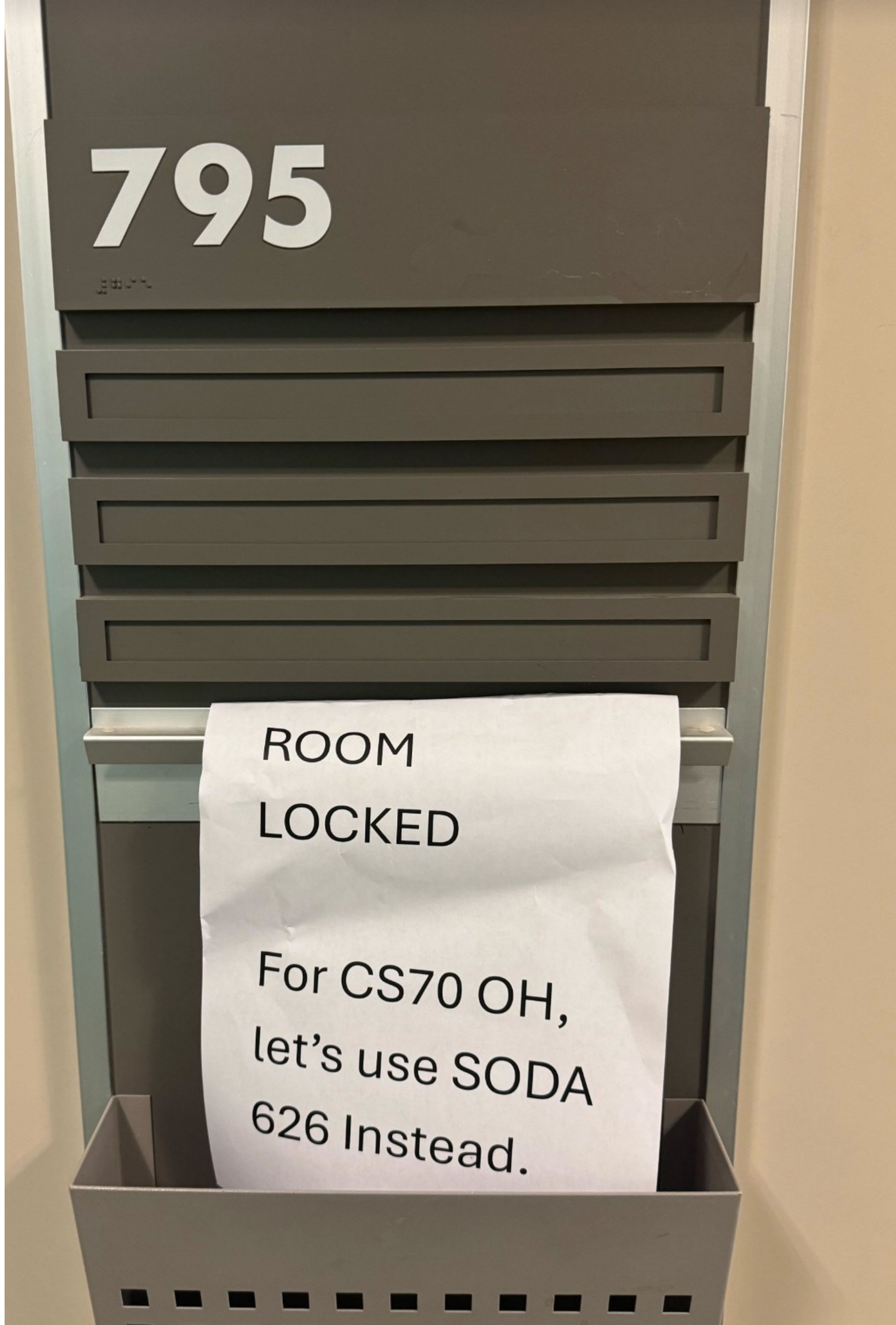
  Jonny’s session (MTuWTh 6pm-7pm) has ~25 students.

  Carolyn’s session (MTuWTh 5pm-6pm) has ~30 students.

- So if you can make it to the other sessions, it’s really helpful if you go to those instead.

Announcements

- Instructor OH
  - Again today 4-5pm at SODA 626
  - Why is it 626? I thought you said 795 last time?



Diagonalization

- Real numbers $\mathbb{R}$ can be defined as countably long decimals.
  - E.g. 0.0023242321……, 131.42345324….., 3.1415926……..
  - Caveat: 1 = 0.999999……..
    3.3 = 3.29999……….

- Is $\mathbb{R}$ countable?

Diagonalization

- Is $\mathbb{R}$ countable?
  - NO!

- Proof.

  Assume $\mathbb{R}$ is countable, then $\mathbb{R}$ is enumerable.

  Take any enumeration,

  0.32123435……

  0.34255235……

  0.12342551……

  0.59285225……

  ……………

Diagonalization

- Is $\mathbb{R}$ countable?
  - NO!   (Equivalent to proving $[0,1)$ is uncountable.)

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  Take any enumeration,

  0.32123435……

  0.36255235……

  0.12642551……

  0.59285225……

  ……………

  0.6776……

We construct a real number not in the list:

If the i-th row's i-th digit is 6, we put 7 in the i-th digit of our number.

Otherwise we put 6.

Diagonalization

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Take any enumeration,

0.32123435……

0.36255235……

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0.59285225……

…………….

0.6776……

Why is not in the list? Proof by contradiction.

Why 6 and 7?

Wait a minute…

- We have seen
  - The set of all strings are countable.
    - This includes every English sentence.
  - The set of all real numbers are uncountable.
    - => Most of the real numbers cannot be described / named / said!
  - A philosophical question: Do they really exist?
    - If a tree falls in a forest….

Recall Induction.

In an induction proof…..

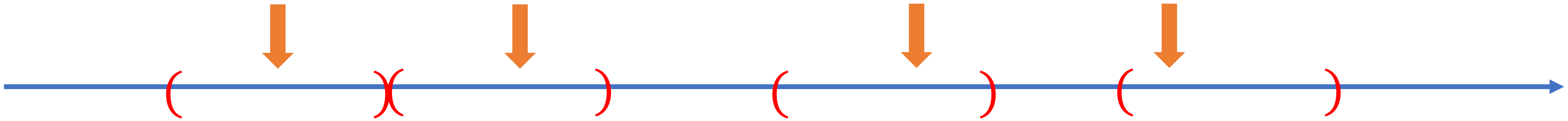
✅ P(0) $\xrightarrow{\text{Inductive step}}$ ✅ P(1) $\xrightarrow{\text{Inductive step}}$ ✅ P(2) $\xrightarrow{\text{Inductive step}}$ ✅ P(n) $\xrightarrow{\text{Inductive step}}$ ✅ P(n+1) $\longrightarrow$ ……………

Why does induction work for natural numbers but not real numbers?

How about… disjoint Intervals?

- Suppose S is a set of disjoint intervals.

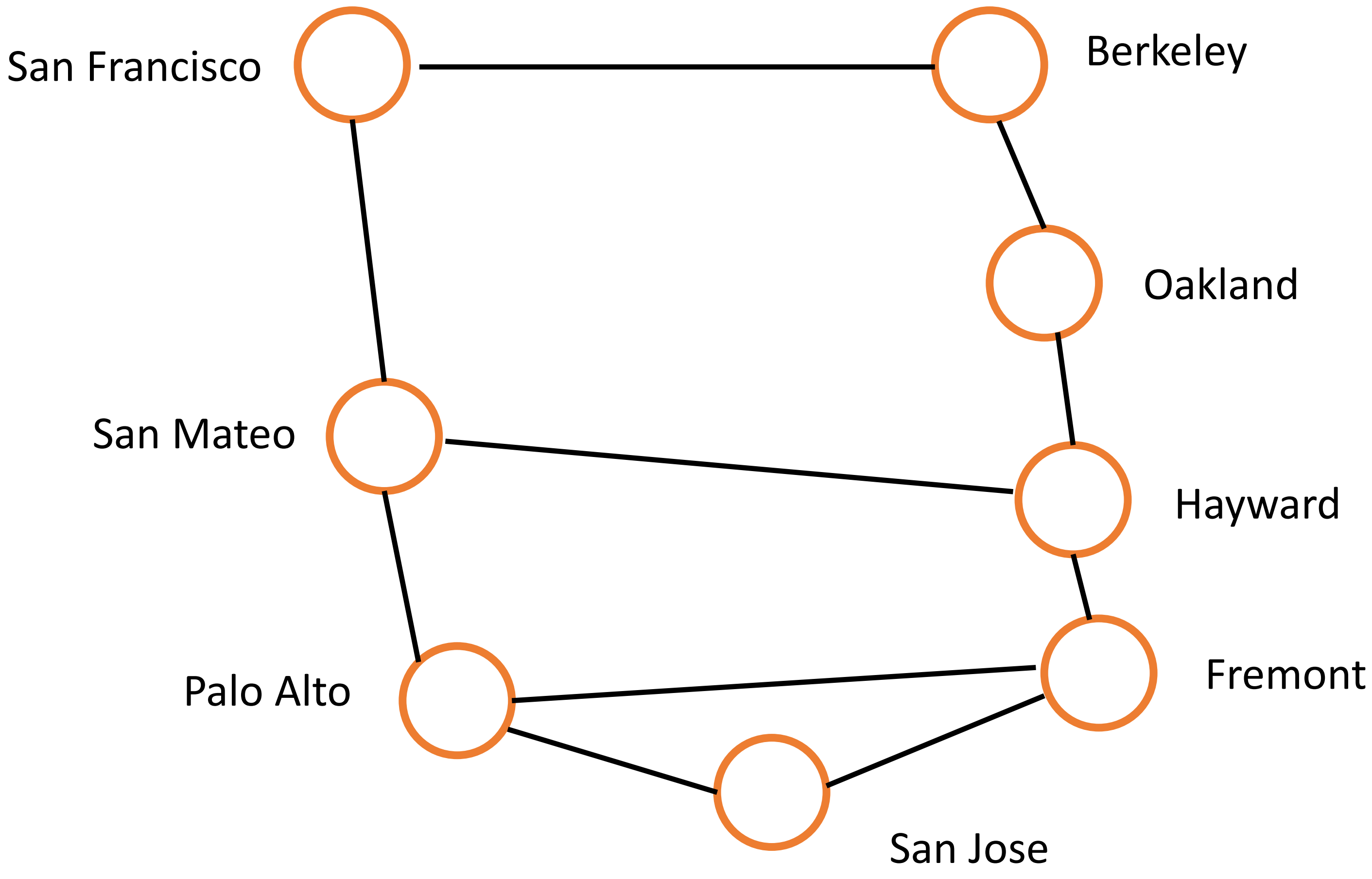
  (e.g. S = $\{(1,2), (\mathrm{e}, \pi), (4, \sqrt{29})$ ….})



S is countable!

Each interval contains at least one rational number.
We can construct injection $f: S \rightarrow \mathbb{Q}$.
Mapping intervals to that rational number. So $|S| \leq |\mathbb{Q}|$.

Lecture 6 & 7: Graphs I & II



Our Plan

- Basic Notions.
  - Graphs
  - Path / walks / cycles.
- Eulerian Tours
  - Existence
  - Algorithm
- Planar graphs
  - Euler's Formula
  - Five coloring theorem
- Different kinds of graphs
  - Complete Graph / Trees / Hypercube
  - Induction on graphs (Bottom-up vs Top-down)

Modeling the real world

- Mathematics is all about abstraction.
- We extract the essential structure that we care about out, and study them.
- Example: Topology

Modeling the real world

- Let us say we have a map of bay area.
- I just want to know how to get from A to B. What is the essential structure?

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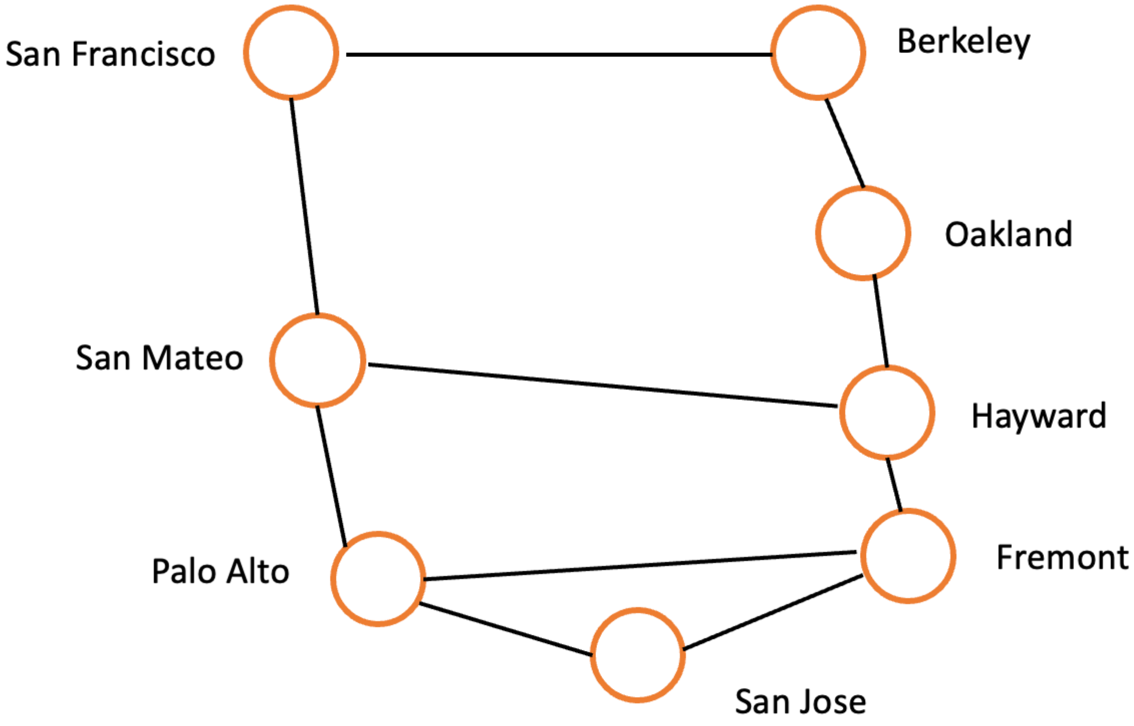
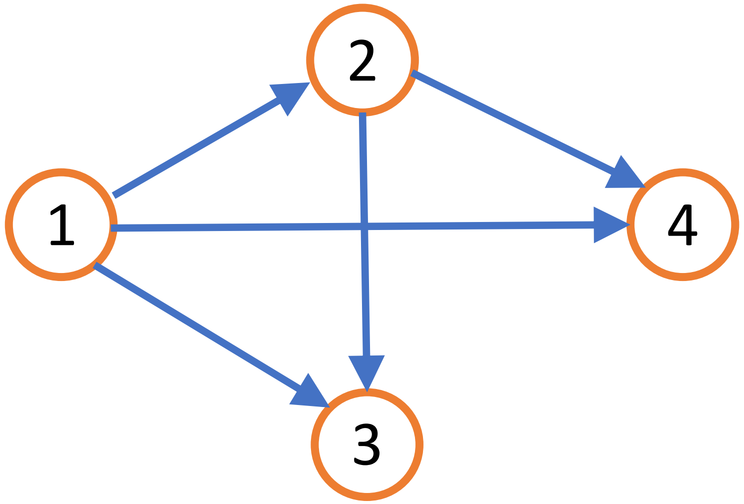


Basic Notions

Definition

A directed graph $G = (V, E)$ consists of vertices V and edges $E \subseteq V \times V$.

For example, $V = \{1,2,3,4\}$, $\mathrm{E} = \{(1,2), (2,3), (1,3), (2,4), (1,4)\}$

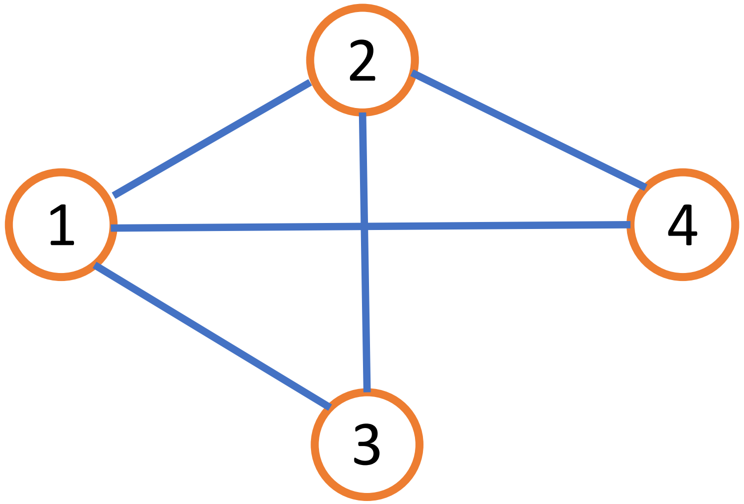


Basic Notions

Definition

An undirected graph $G = (V, E)$ consists of vertices V and edges E which is set of unordered sets of two vertices.

For example, $V = \{1,2,3,4\}$, $\mathrm{E} = \{\{1,2\},\{2,3\},\{1,3\},\{2,4\},\{1,4\}\}$



Path / Walk / Cycle / Tour

Definition

A path is a sequence of connected edges

$$\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{k-1}, v_k\}.$$

with distinct vertices $v_1, v_2, \dots, v_k$.

We say this path is from from v_1 to v_k: $v_1 \to v_2 \to \cdots \to v_k$.

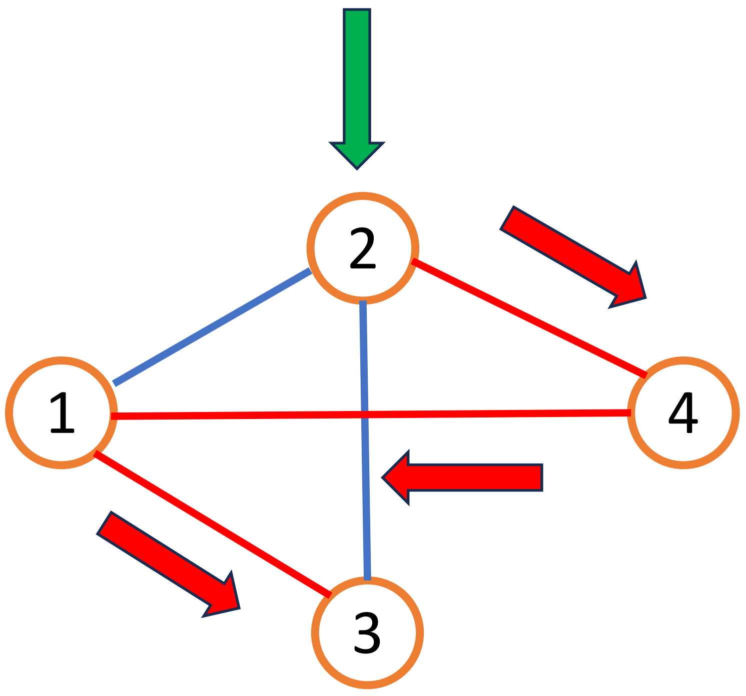
Definition

A walk is a sequence of connected edges

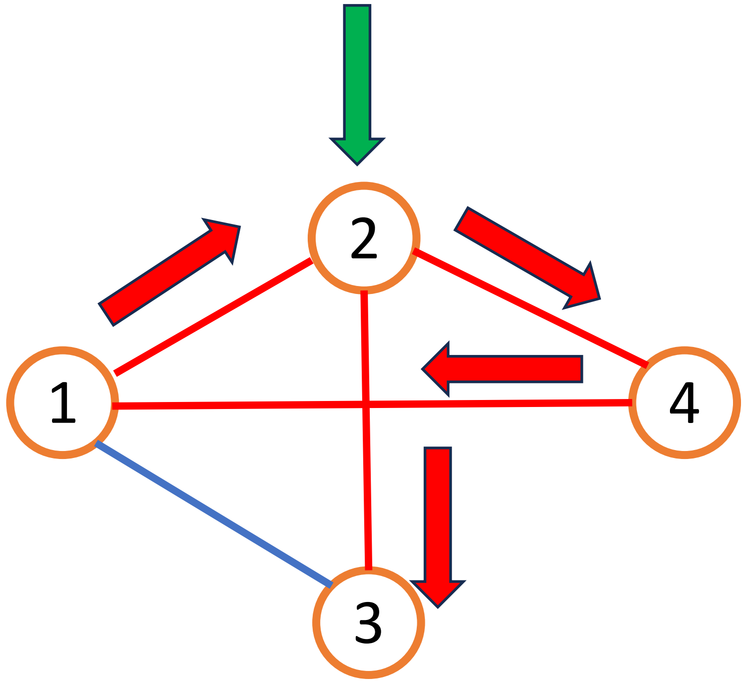
$$\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{k-1}, v_k\}.$$

with possibly repeating vertices $v_1, v_2, \dots, v_k$.

Path / Walk / Cycle / Tour



path



walk

Path / Walk / Cycle / Tour

Definition

A cycle is a sequence of connected edges

$$\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{k-1}, v_k\}.$$

with distinct vertices $v_1, v_2, \ldots, v_{k-1}$, and $v_k = v_1$.

Definition

A tour is a sequence of connected edges

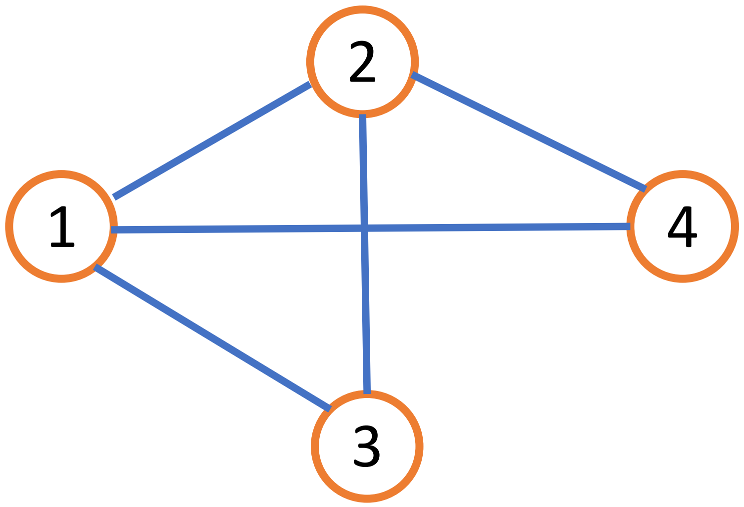
$$\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{k-1}, v_k\}.$$

with possibly repeating vertices $v_1, v_2, \ldots, v_{k-1}$, and $v_k = v_1$.

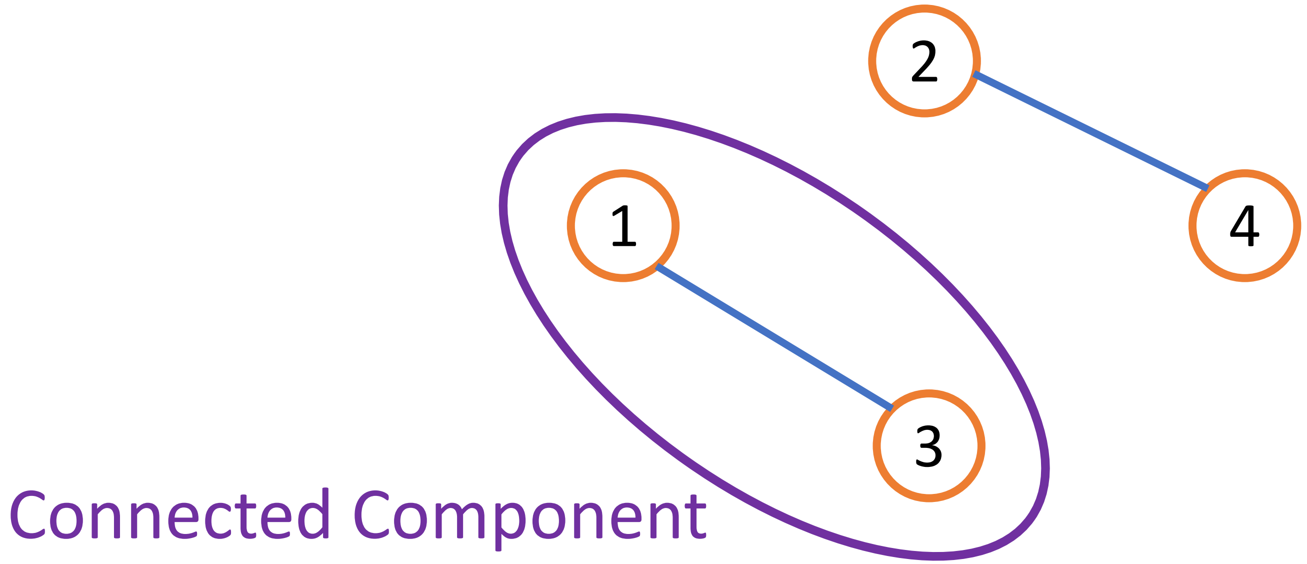
Connectivity

Definition

We say an undirected graph G is connected, if there exists a path between any two vertices (connecting them).



connected



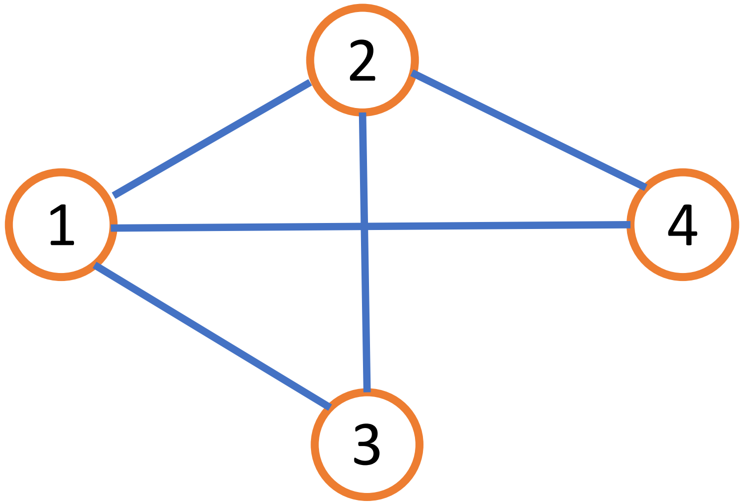
disconnected

Degree

Definition

In undirected graph, the degree of a vertex is the number of edges connects to it.

For example, here $\deg(2) = 3, \deg(4) = 2$.

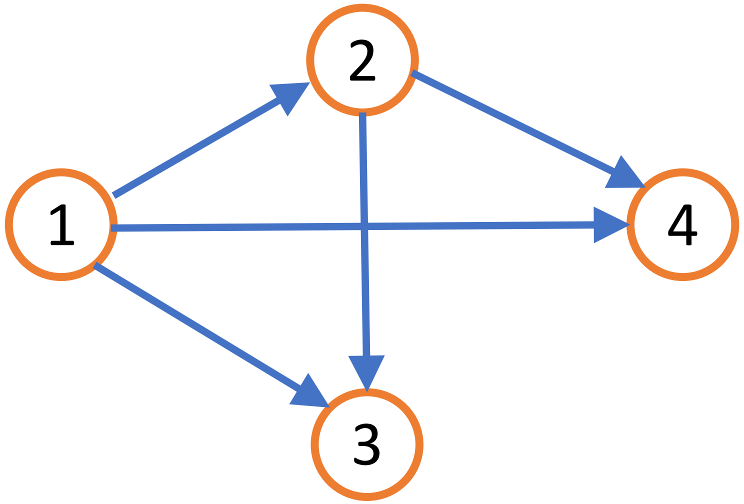


Degree

Definition

In directed graph, the in-degree of a vertex is the number of edges entering it. The out-degree of a vertex is the number of edges leaving it.

For example, here $\text{in−deg}(2) = 1, \text{out−deg}(2) = 2$.



Degree & Sum of degree

Lemma (Handshaking Lemma)

In undirected graph,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Induction on graphs

Lemma (Handshaking Lemma)

For all undirected graph,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof (By induction)

Base Case: When |E|=0, there is no edge. This is true.

Induction Hypothesis: Suppose this is true for all graphs with $\leq$ e-1 edges.

Inductive Step: We want to prove it for graphs with e edges.

We remove one edge from the graph. The degree at both end points -1.

………

Build-up error

Note in the induction, we start with a graph with e edges and remove 1 edge from it.

What if we start with a graph with $e-1$ edges and add 1 edge to it?

Build-up error

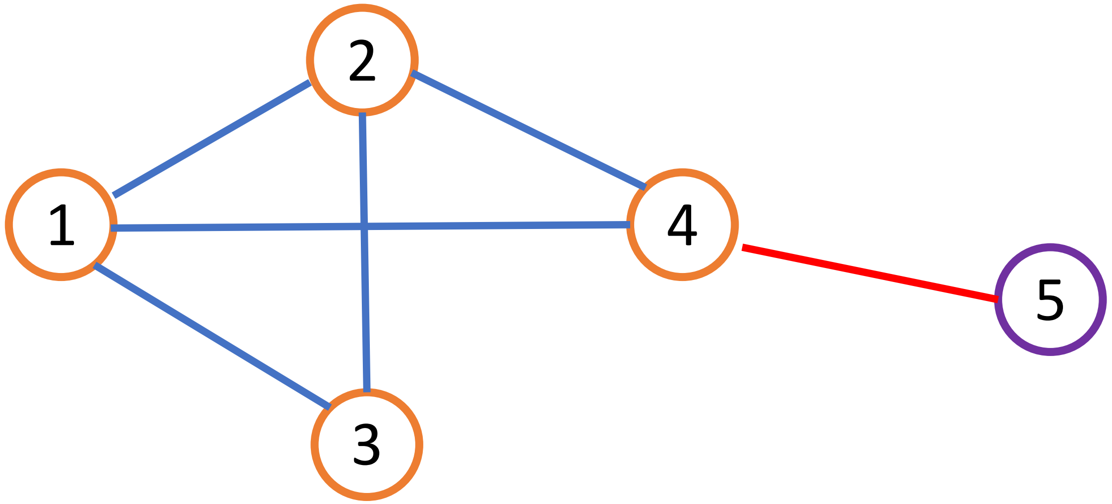
Base Case: When |E|=0, there is no edge. This is true.

Induction Hypothesis: Suppose this is true for all graphs with $\leq$ e-1 edges.

Inductive Step: We want to prove it for graphs with e edges.

We take a graph with e -1 edges and add a new edge between that graph and a new vertex.

Sum of degree +2, #edge + 1. Thus equality still hold.



Is this a valid proof?

Recall Induction.

In an induction proof.....

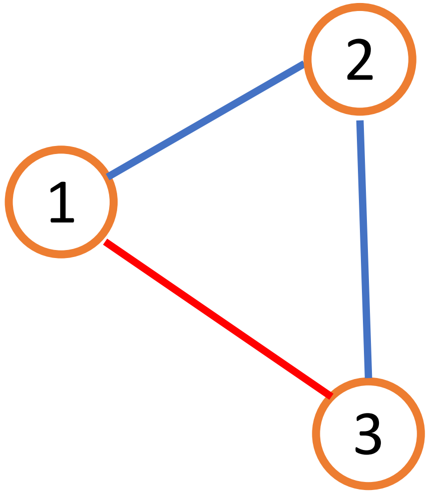
✅ P(0) —Inductive step→ ✅ P(1) —Inductive step→ ✅ P(2) —Inductive step→ ✅ P(n) —Inductive step→ ✅ P(n+1) ——→ ....

Why does induction work for natural numbers but not real numbers?

Because we can reach all natural number!

Build-up Error

- If you always add a new edge between current graph and a new vertex.



You can't even reach this cycle.

Example: Friend Chain at a Conference

Problem.

You and John are both attending a conference, but you don’t know each other. Even worse, both you and john each only have one friend in that conference. All other people in the conference have even number of friends in the conference (including zero).

Prove that there must be a friend chain, you – person1 – person2 – ⋯ – person k – John, connecting you and john. Each adjacent two people on the chain are friends.

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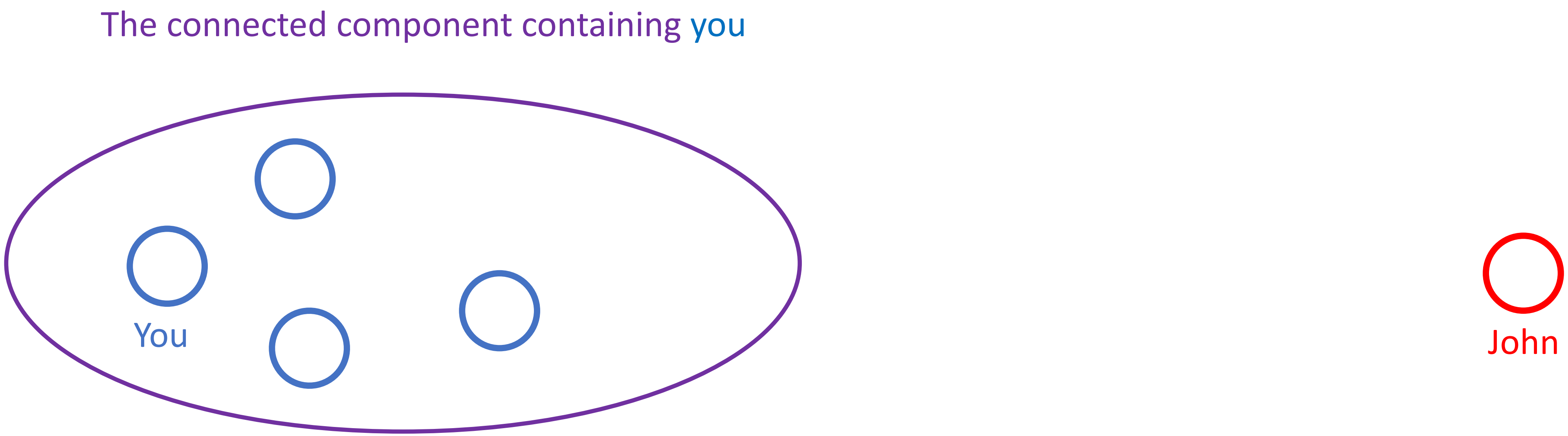

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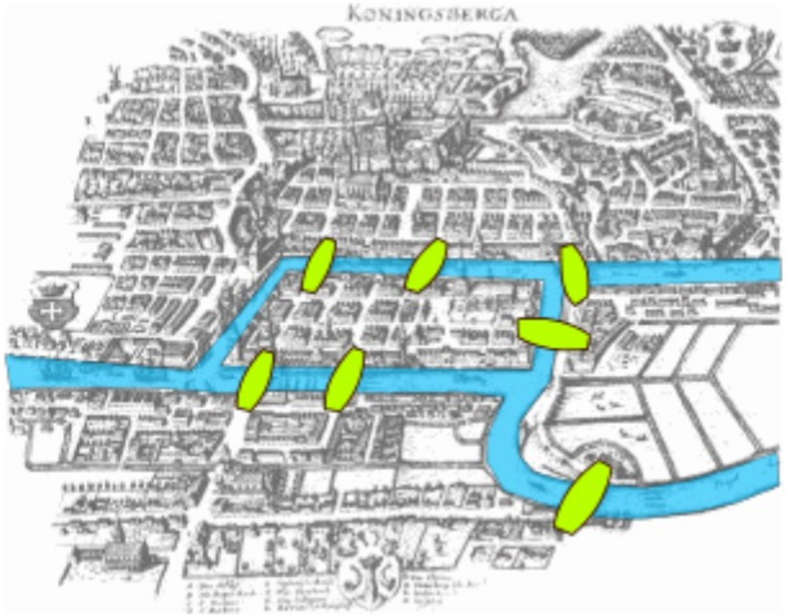
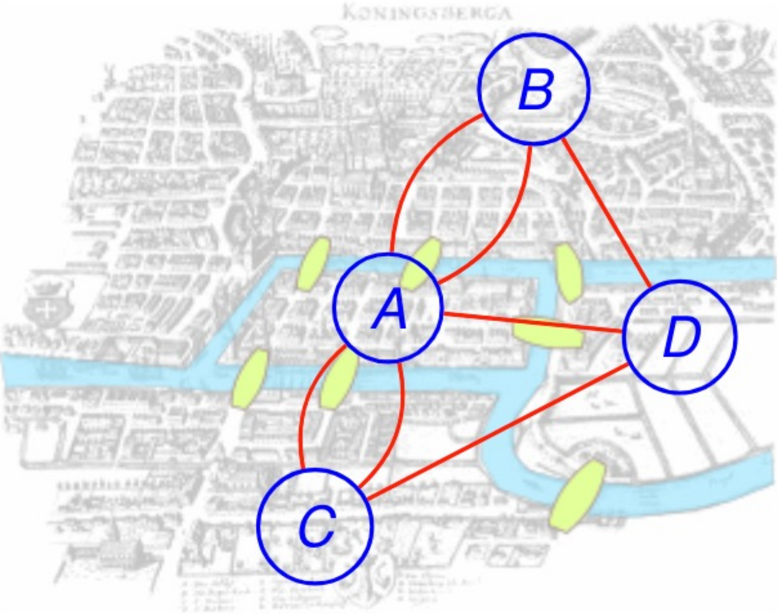
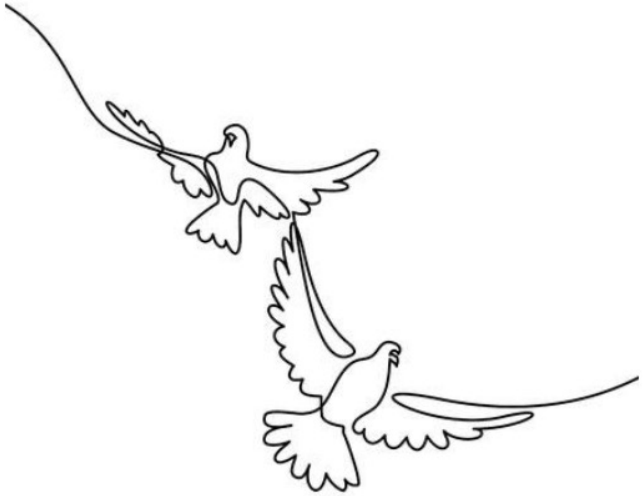
Seven-Bridge Problem

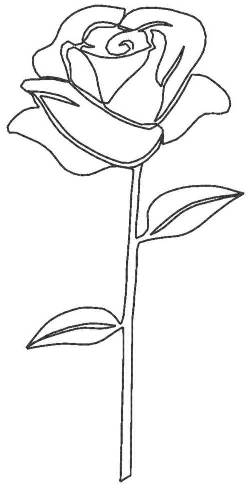
“Konigsberg bridges” by Bogdan Giuşcă - License.





Drawing with one line





Eulerian Tours

Definition

A tour is a sequence of connected edges

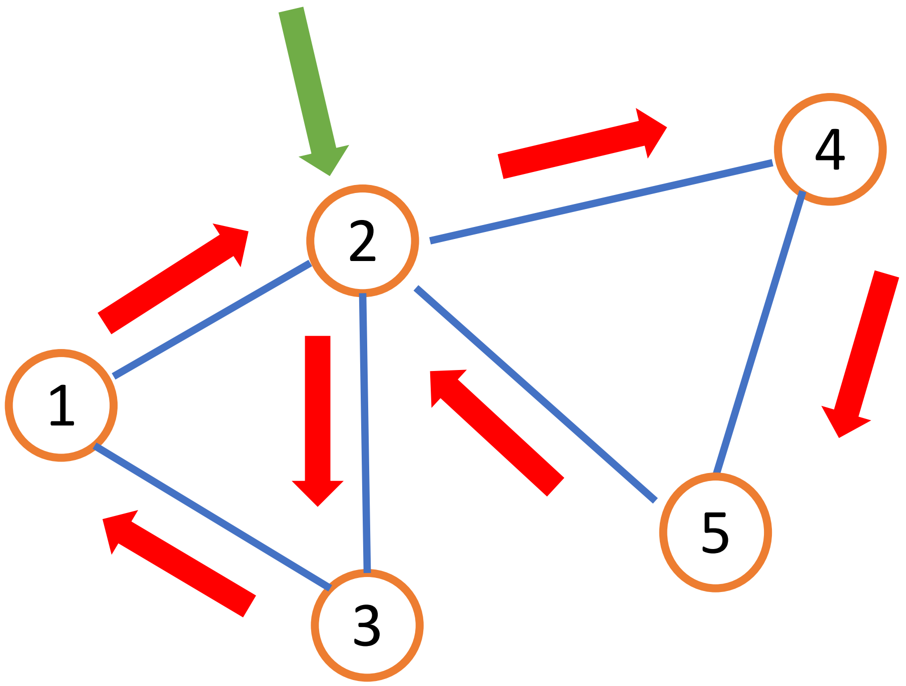
$$\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{k-1}, v_k\}.$$

with possibly repeating vertices $v_1, v_2, \dots, v_{k-1}$, and $v_k = v_1$.

Definition

An Eulerian tour is a tour that traverses every edge exactly once in an undirected graph.

Example



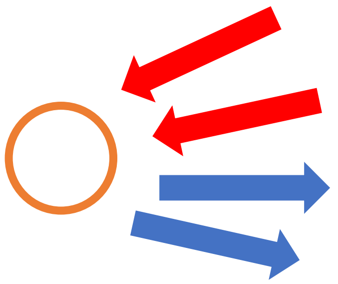
Eulerian Tours

Theorem

An undirect graph has an Eulerian tour if and only if each vertex has even degree and the graph is connected.

Proof (only if):

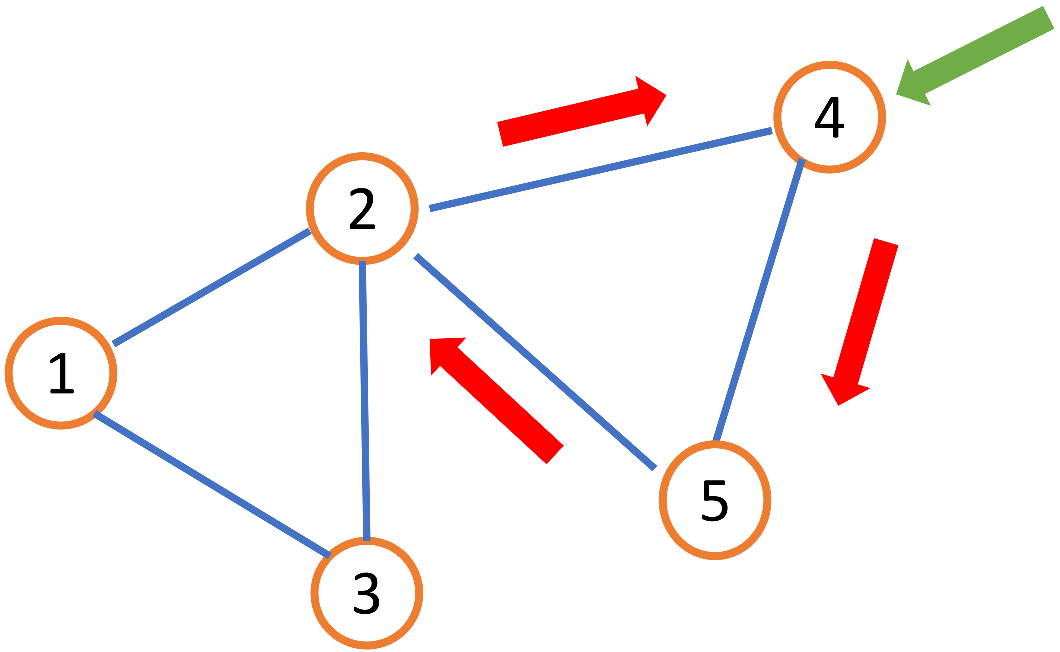
- If a vertex have odd degree, you cannot exit it after every time you enter it.
- If the graph is not connected, there is no tour through all edge.



Naïve algorithm

FINDTOUR(G, s)

Starting from s, follow arbitrary untraveled edge, until get stuck.



It always return a tour. But may not be a Eulerian tour.

Full algorithm

FINDTOUR(G, s)

Starting from s, follow arbitrary untraveled edge, until get stuck.

Euler(G)

T = FINDTOUR(G, s) for an arbitrary vertex s in G

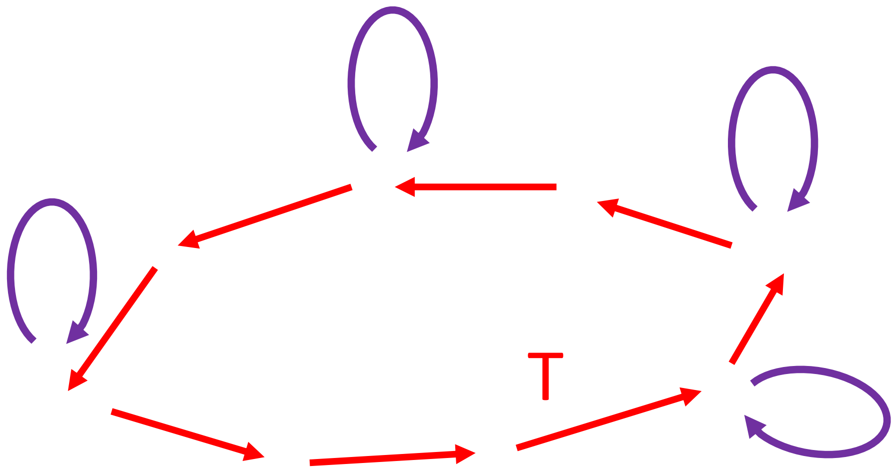
Remove T to get G'.

For every connected component G_i' in G',

Euler(G_i')

Splice together the Eulerian tours and T.

Eulerian tours



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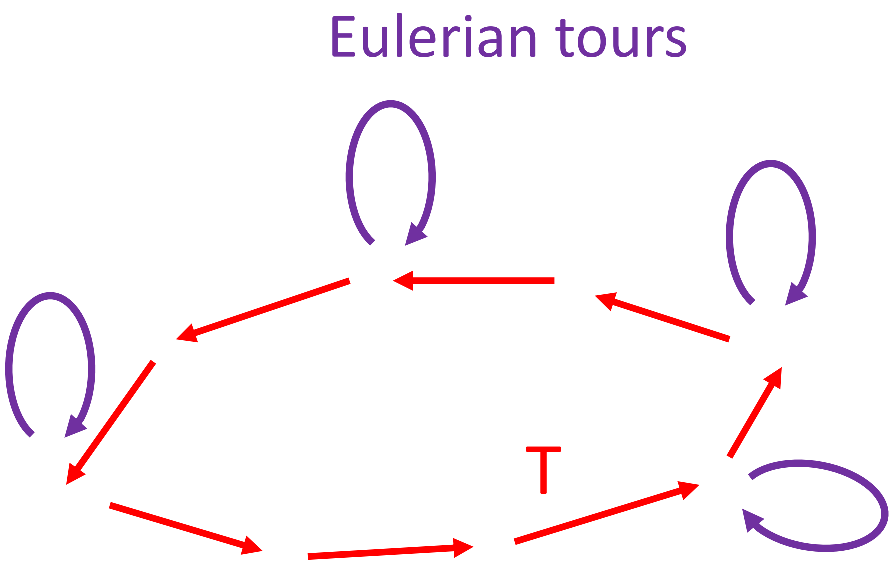
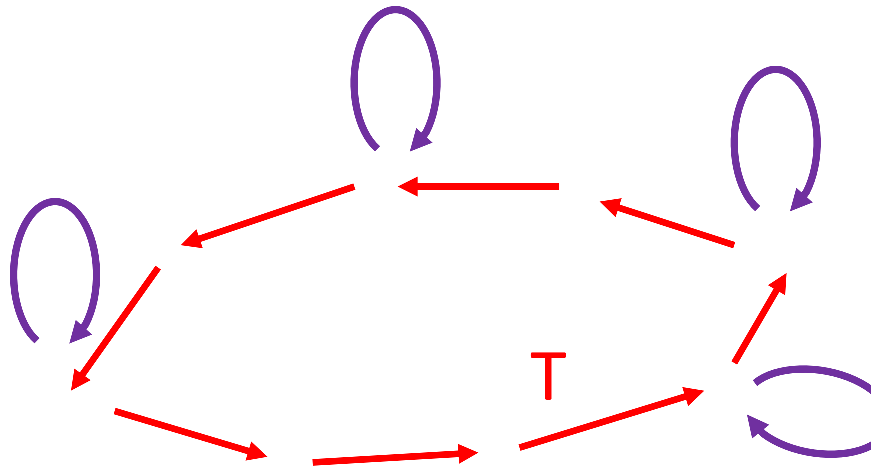


Why it works?

Proof by Induction (sketch)
Base Case: When it is a connected graph with no edge (a single vertex).
Induction Hypothesis: Suppose it works for all graphs with <= e – 1 edges.

Inductive Step: For graph G with e edges, remove tour T.
For each connected component, it must have <= e – 1 edges, apply induction hypothesis. The algorithm must be able to find a Eulerian tour for each component.

Finally, show that splice works.



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