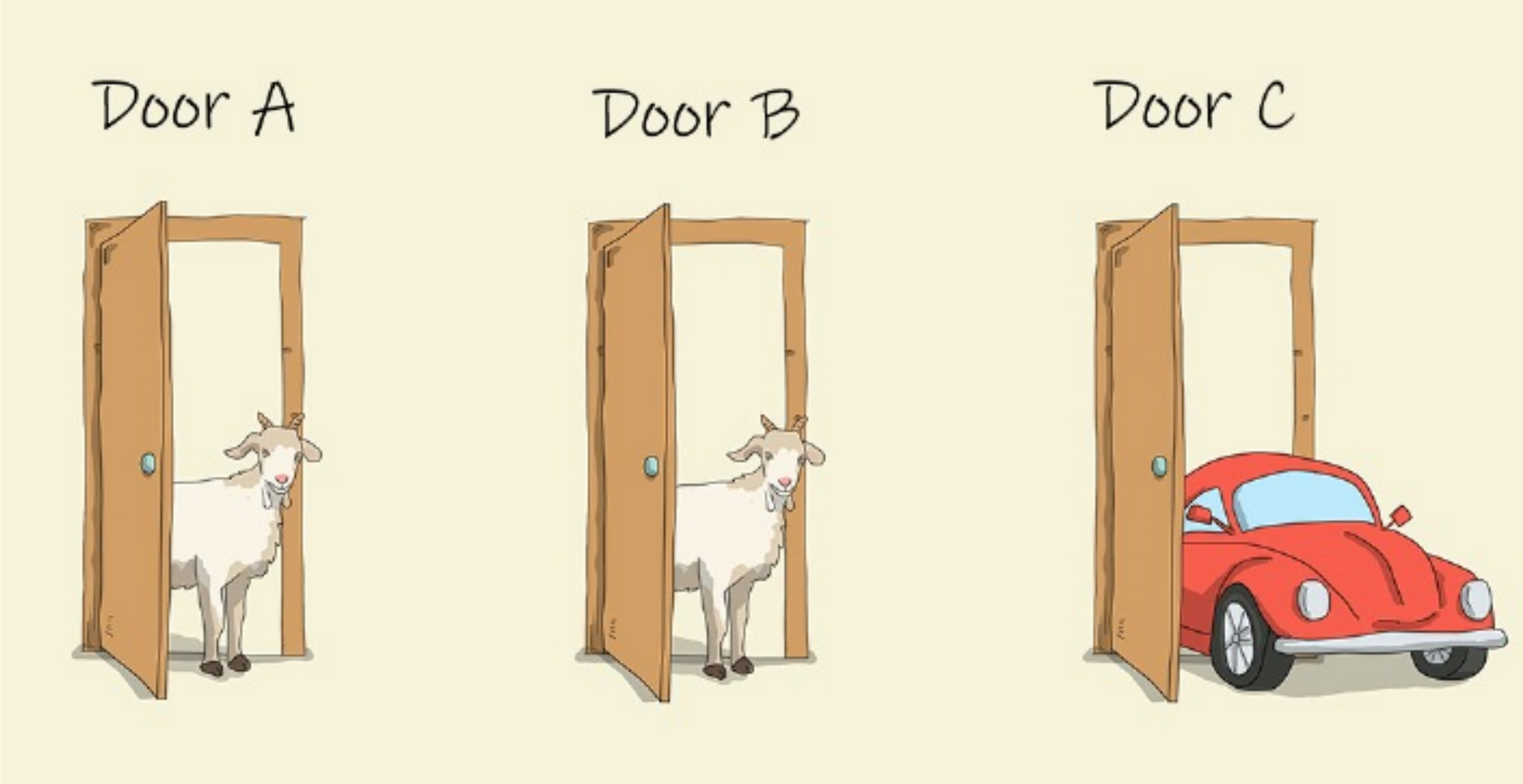
Lecture 20: Gems of probability theory

Sit back and relax.

Monty Hall Problem

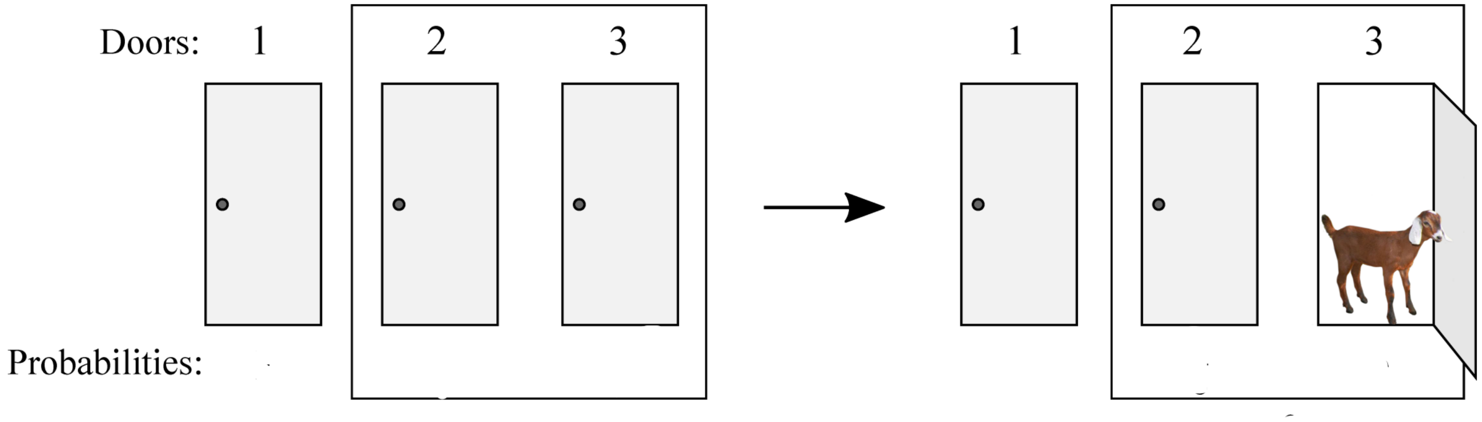
There are three doors. Behind one there is a car. Behind the other two are just goats.



Monty Hall Problem

There are three doors. Behind one there is a car. Behind the other two are just goats.

You choose one door (not opened). Then the host opens a door in the remaining two doors that has a goat behind it.



Then the host asks you whether you'd like to switch.

Monty Hall Problem

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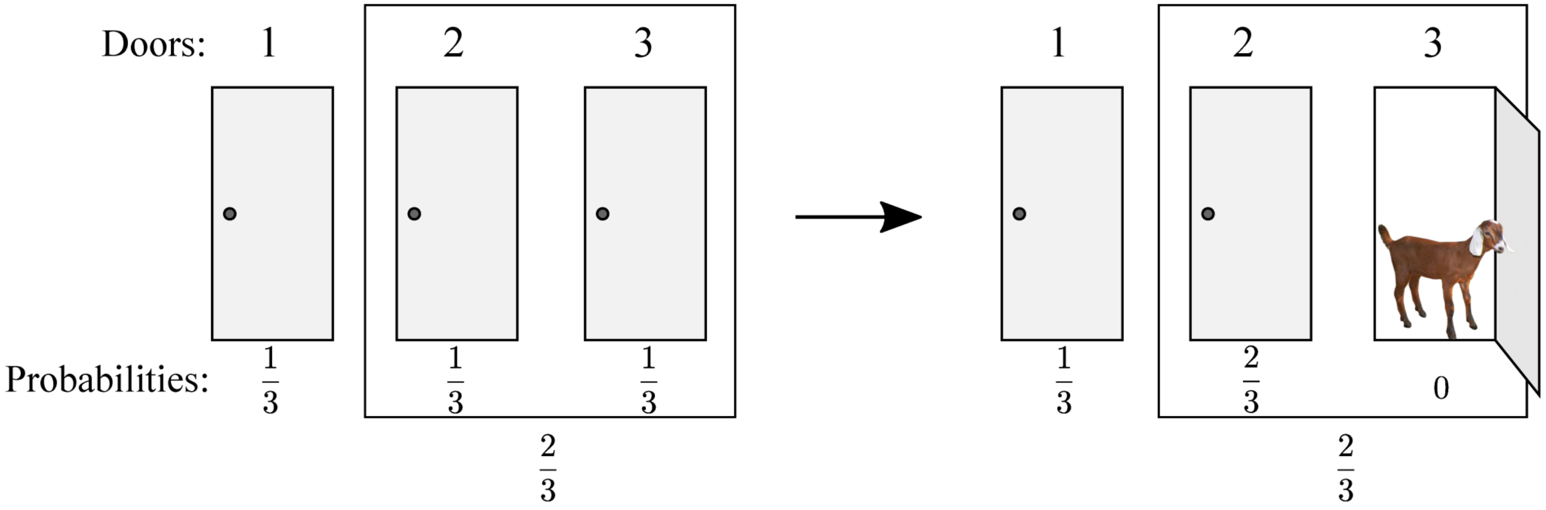
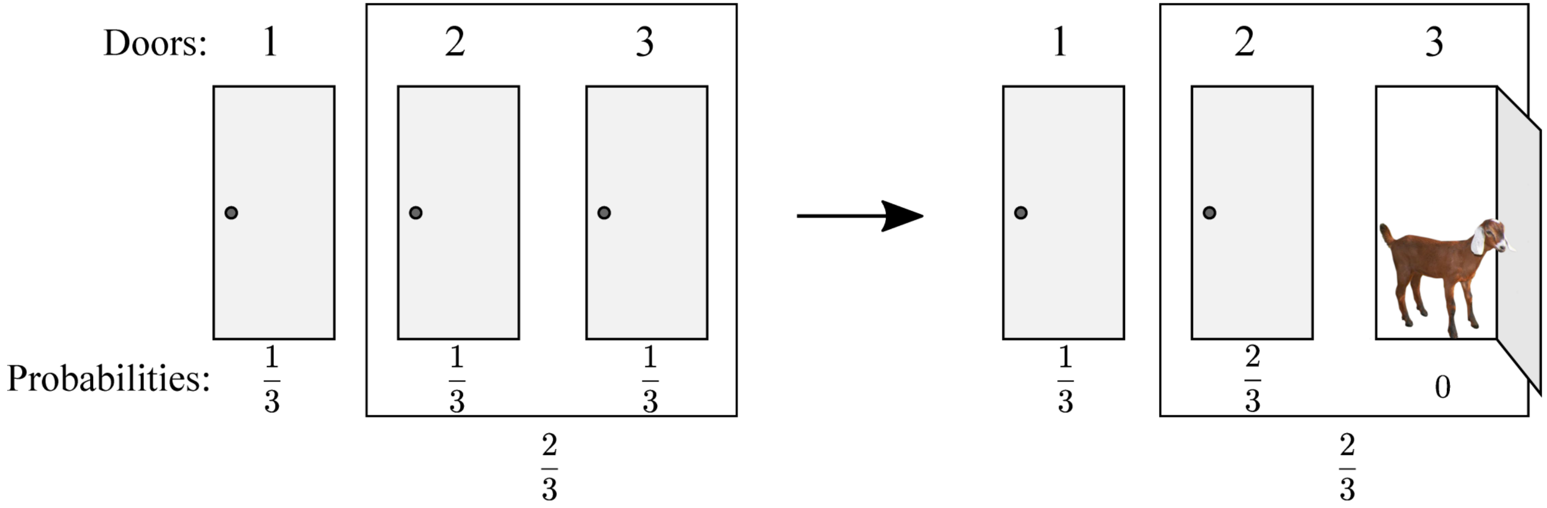


The smart thing to do: Always switch!

Monty Hall Problem



$$\mathbb{P}(\text{car behind the door you first choose}) = \frac{1}{3}$$

$$\mathbb{P}(\text{car behind the other two doors}) = \frac{2}{3}$$

Opening a door does not change these two probabilities!

Chance of Cancer

Alice took a test for a rare kind of cancer and got a positive result.

A priori:

P(cancer) = 1/1000,000

Test:

P(positive | cancer) = 1,   P(negative | cancer) = 0

P(positive | not cancer) = 1%, P(negative | not cancer) = 99%.

What is the chance that Alice got cancer?

Chance of Cancer

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P(positive | cancer) = 1,   P(negative | cancer) = 0

P(positive | not cancer) = 1%, P(negative | not cancer) = 99%.

$$P(\text{cancer} \mid \text{positive}) = \frac{P(\text{positive} \mid \text{cancer}) \cdot P(\text{cancer})}{P(\text{positive} \mid \text{cancer}) \cdot P(\text{cancer}) + P(\text{positive} \mid \text{not cancer}) \cdot P(\text{not cancer})}$$

Chance of Cancer

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P(cancer | positive) = $\frac{1*1/1000{,}000}{1*1/1000{,}000+0.01\cdot(1-1/1000{,}000)} \approx \frac{1*1/1000{,}000}{0.01}$ = 1/10,000

Debate: People vs. Collins

What witness said

Trial [edit]

After a mathematics instructor testified about the multiplication rule for probability, though ignoring conditional probability, the prosecutor invited the jury to consider the probability that the accused (who fit a witness's description of a black male with a beard and mustache and a Caucasian female with a blond ponytail, fleeing in a yellow car) were not the robbers, suggesting that they estimated the probabilities as:

Black man with beard	1 in 10
Man with mustache	1 in 4
White woman with pony tail	1 in 10
White woman with blond hair	1 in 3
Yellow motor car	1 in 10
Interracial couple in car	1 in 1,000

The jury returned a guilty verdict.[1]

What prosecutor argued

Think of P(guilty | match description) as P(cancer | test).

The prosecutor only argued P(match description | not guilty) is small.

Simpson's Paradox

Gender bias study of grad admission of Berkeley in Fall of 1973

50 years ago here! (Table from Wikipedia)

	All		Men		Women	
	Applicants	Admitted	Applicants	Admitted	Applicants	Admitted
Total	12,763	41%	8,442	44%	4,321	35%

People thought more men were admitted.

Simpson's Paradox

Gender bias study of grad admission of Berkeley in Fall of 1973

Department	All		Men		Women	
	Applicants	Admitted	Applicants	Admitted	Applicants	Admitted
A	933	64%	**825**	62%	108	82%
B	585	63%	**560**	63%	25	68%
C	918	35%	325	37%	**593**	34%
D	792	34%	417	33%	375	35%
E	584	25%	191	28%	**393**	24%
F	714	6%	373	6%	341	7%
Total	**4526**	**39%**	**2691**	**45%**	**1835**	**30%**

More women was admitted in each department!

Simpson's Paradox

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More men applied to Department A/B that are easier to get into.
More women applied to Department C/D that are harder to get into

Envelope Paradox

Say I have two envelopes that contain $$$

One contains twice the money of the other one and I randomly swapped two.

Strategy 1: Pick one envelope, get x dollars.

Strategy 2: Switch to other one,

with ½ probability, get $x/2$ dollars.

with ½ probability, get $2x$ dollars.

In expectation, $\frac{1}{2} \cdot \frac{x}{2} + \frac{1}{2} \cdot 2x = 1.5\ x$

What's wrong?

Envelope Paradox

The missing part:

What is the distribution for x?

If it is not uniform / uniform over only a bounded set, the value of x reaveals information about which envelope it is. Conditioning on that, the two probabilities are no longer ½ and ½ .

But uniform distribution over all real numbers / all natural numbers has infinite expectation. So 1.5 * inf = inf.

The waiting time paradox

We model the arrival of a bus as an infinitely long coin flip sequence.

… … …  … … …

The t-th coin: At the t-th minute, is there a bus arriving? (bus arrival ⇔ head)

Suppose a bus arrives at each minute with probability p.

The waiting time paradox

We model the arrival of a bus as an infinitely long coin flip sequence.

… … …  … … …

How long is the expected time gap between two buses?

E[gap] = 1/p.

If I arrive at a random minute, how many minutes in expectation do I have to wait?

- E[gap / 2] = 1/2p ?
- E[more flips till head] = 1/p ?

The waiting time paradox

We model the arrival of a bus as an infinitely long coin flip sequence.

How long is the expected time gap between two buses?

E[gap] = 1/p. → The length of a uniformly selected gap. $\frac{gap_1 + gap_2 + \cdots + gap_n}{n}$

If I arrive at a random minute, how many minutes in expectation do I have to wait?

- E[gap / 2] = 1/2p ? → The half length. $\frac{gap_1 + gap_2 + \cdots + gap_n}{2n}$
- E[more flips till head] = 1/p ? → You are more likely to arrive during longer gaps!

The waiting time paradox

We model the arrival of a bus as an infinitely long coin flip sequence.

How long is the expected time gap between two buses?

E[gap] = 1/p. → The length of a uniformly selected gap. $\frac{gap_1 + gap_2 + \cdots + gap_n}{n}$

If I arrive at a random minute, how many minutes in expectation do I have to wait?

- E[gap / 2] = 1/2p ? → The half length. $\frac{gap_1 + gap_2 + \cdots + gap_n}{2n}$
- E[more flips till head] = 1/p ? → $\sum_i \frac{gap_i}{gap_1 + gap_2 + \cdots + gap_n} \cdot gap_i$

The waiting time paradox

We model the arrival of a bus as an infinitely long coin flip sequence.

So you usually need to wait 2x time for buses.

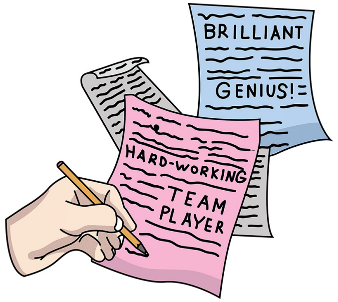
Bayesian Persuasion

The next example is purely fictional and has nothing to do with reality.

Bayesian Persuasion

What should Jelani do to maximize # of his students being admitted?

Say prof. Jelani Nelson Is writing rec letters for his students.



EECS

A Priori

1/3 of his students are outstanding.

2/3 of his students are ordinary.

He can choose to write Strong/Normal letter for each student.

EECS admits a student if P(student is outstanding | letter) ≥ 1/2.

Bayesian Persuasion

A Priori

$1/3$ of his students are outstanding.

$2/3$ of his students are ordinary.

He can choose to write Strong/Normal letter for each student.

EECS admits a student if P(student is outstanding | letter) $\geq 1/2$.

Strategy 1: Super honest Jelani

Write Strong letter for outstanding students.

Write Normal letter for ordinary students.

P(student is outstanding | Strong letter) = 1

P(student is outstanding | Normal letter) = 0

$1/3$ of his students will be admitted.

Bayesian Persuasion

A Priori

1/3 of his students are outstanding.

2/3 of his students are ordinary.

He can choose to write Strong/Normal letter for each student.

EECS admits a student if P(student is outstanding | letter) ≥ 1/2.

Strategy 2: Super nice Jelani

Write Strong letter for every students.

P(student is outstanding | Strong letter) = 1/3

None of his students will be admitted.

Bayesian Persuasion

A Priori

1/3 of his students are outstanding.

2/3 of his students are ordinary.

He can choose to write Strong/Normal letter for each student.

EECS admits a student if P(student is outstanding | letter) $\geq$ 1/2.

Strategy 3: Smart Jelani

Write Strong letter for outstanding students.

Write Strong letter with probability ½ for ordinary students.

Write Normal letter with probability ½ for ordinary students.

P(student is outstanding | Strong letter) = $\frac{\frac{1}{3}*1}{\frac{1}{3}*1+\frac{2}{3}*\frac{1}{2}} = \frac{1}{2}$

$\frac{1}{3} * 1 + \frac{2}{3} * \frac{1}{2} = \frac{2}{3}$ of his students will be admitted.

Bayesian Persuasion

Bayesian persuasion

E Kamenica, M Gentzkow - American Economic Review, 2011 - aeaweb.org Paperpile

… If so, what is the optimal way to **persuade**? These questions are of substantial economic … , attempts at **persuasion** command a sizable share of our resources. **Persuasion**, as we will …

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A very influential paper in econ! 2.8k citations in ~10 years.

Zero Knowledge Proof

How do you prove to me you know how to solve this sudoku without revealing its answer at all?

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Zero Knowledge Proof

sudoku rule: 1. each row/column contains each of 1,2,3 ... , 9 exactly once
` 2. each 3x3 block contains each of 1,2,3 ... , 9 exactly once

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Zero Knowledge Proof

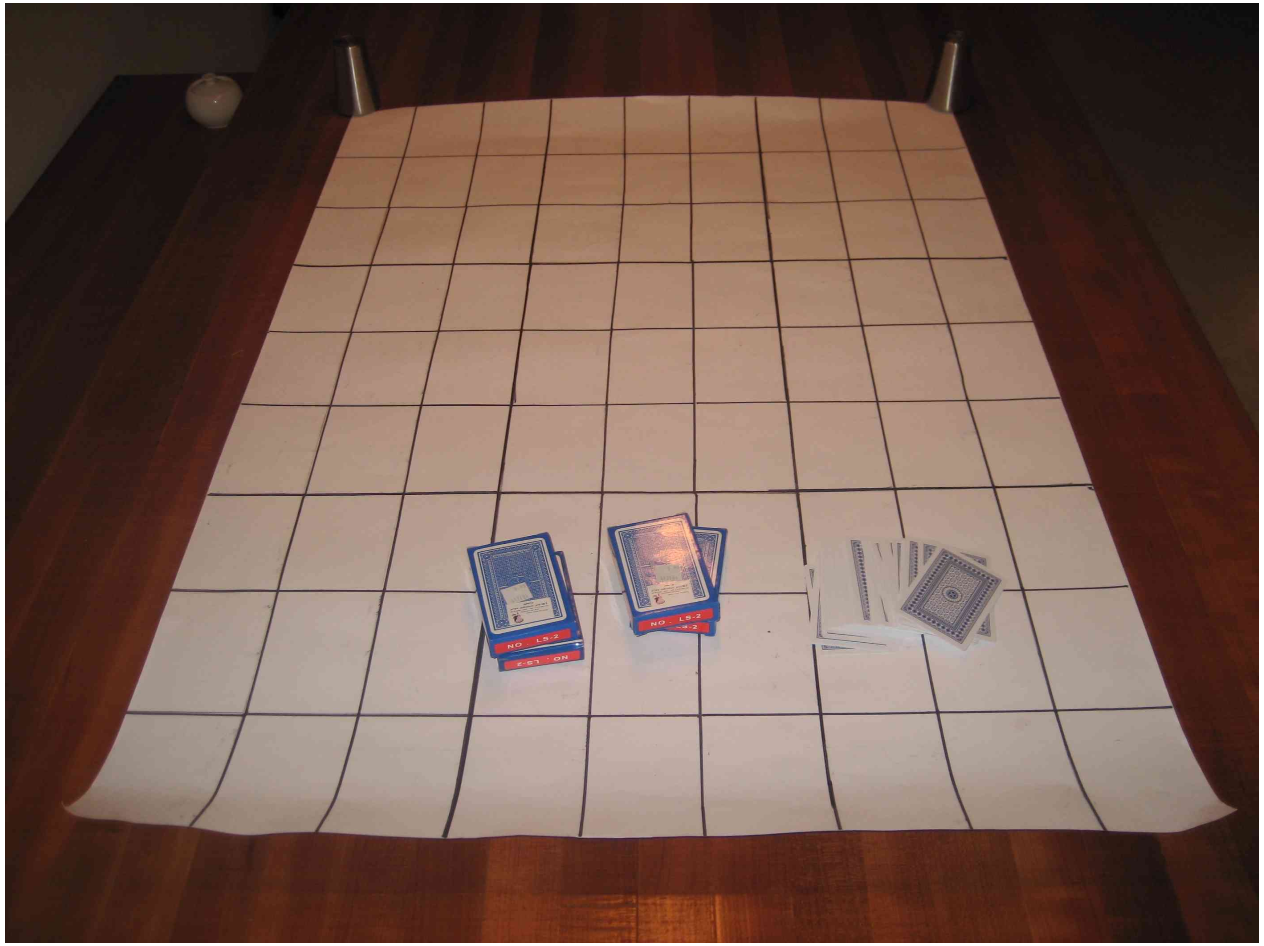
Zero-knowledge Proof

You as the prover:

1. Find enough poker cards and use their number to represent the answer.
2. For empty slots, put the card with the number upside down.
   For fixed slots, put the card facing up.

Zero Knowledge Proof

Zero-knowledge Proof (Photo by Moni Noar)



Zero Knowledge Proof

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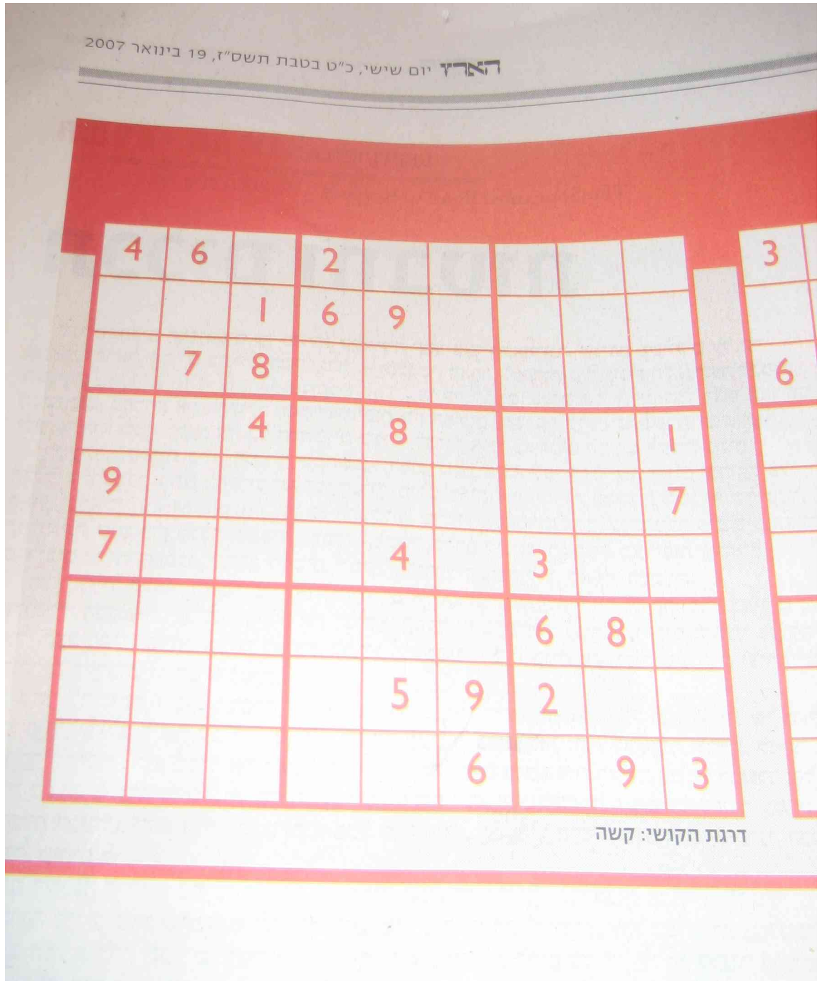

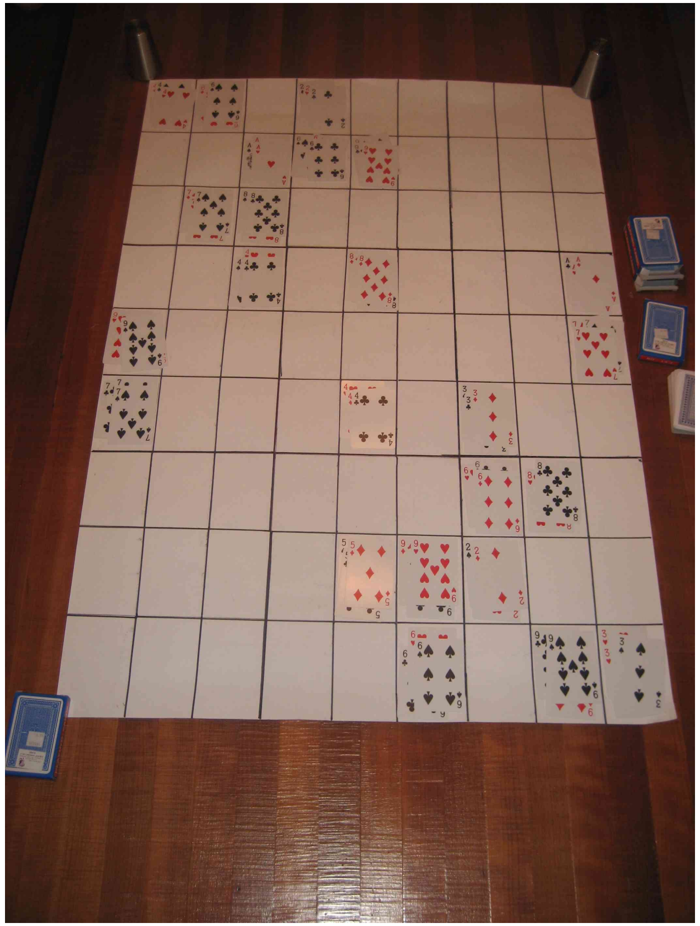

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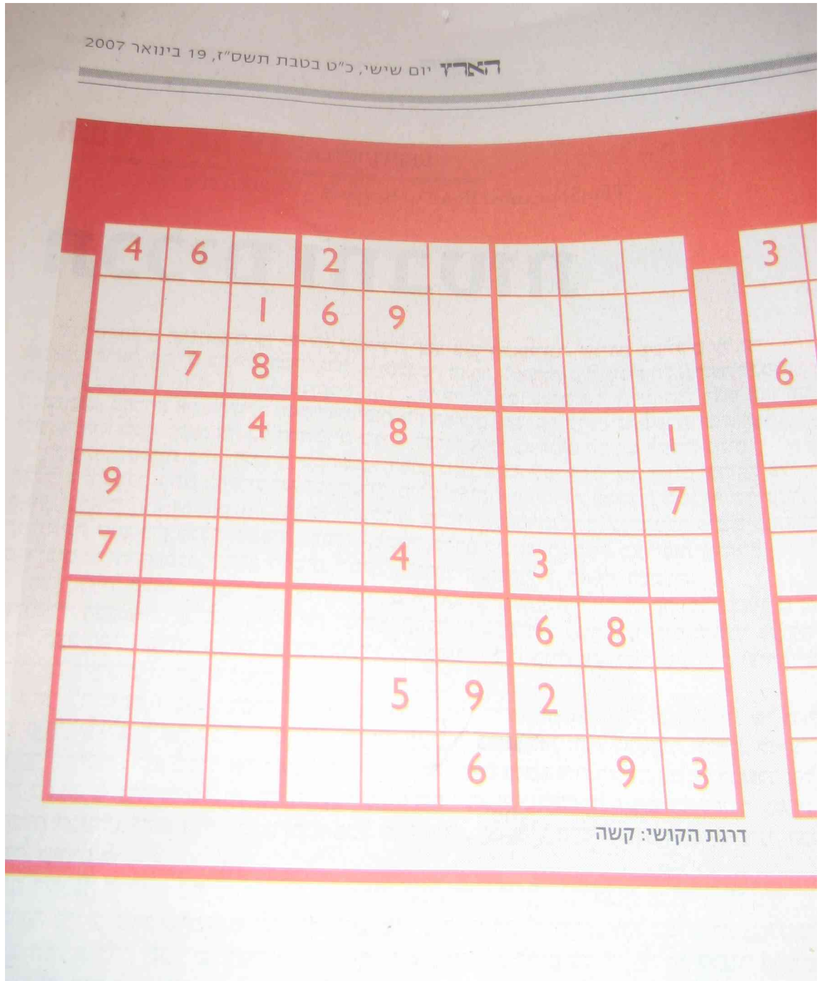

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Zero-knowledge Proof

You as the prover:

1. Find enough poker cards and use their number to represent the answer.
2. For empty slots, put the card with the number upside down.
   For fixed slots, put the card facing up.

Me as the verifier:

1. Check if all fixed slots have the same number as the problem.
2. Randomly select a row / column / 3x3 block, and collect all cards in those slots while keeping the upside-down cards secret.
3. Shuffle the cards without looking at them. Then check if they are 1 – 9.

Zero Knowledge Proof

Zero-knowledge Proof

If the prover is honest, the verifier will only see 1-9.

P(answer = a | seeing 1-9) = P(answer = a)   It reveals no information at all!

Me as the verifier:

1. Check if all fixed slots have the same number as the problem.
2. Randomly select a row / column / 3x3 block, and collect all cards in those slots while keeping the upside-down cards secret.
3. Shuffle the cards without looking at them. Then check if they are 1 – 9.

Zero Knowledge Proof

Zero-knowledge Proof

If the prover is honest, the verifier will only see 1-9.

P(answer = a | seeing 1-9) = P(answer = a)   It reveals no information at all!

Me as the verifier:

9 rows + 9 columns + 9 blocks.

If the prover is lying, catch them with prob $\geq 1/18$.

If we repeat 100 times, catch them with probability $1-\left(1-\frac{1}{18}\right)^{100}=99.6\%$

Zero Knowledge Proof

Zero-knowledge Proof

Invented by Goldwasser, Micali, Rackoff in the 80s @ Berkeley

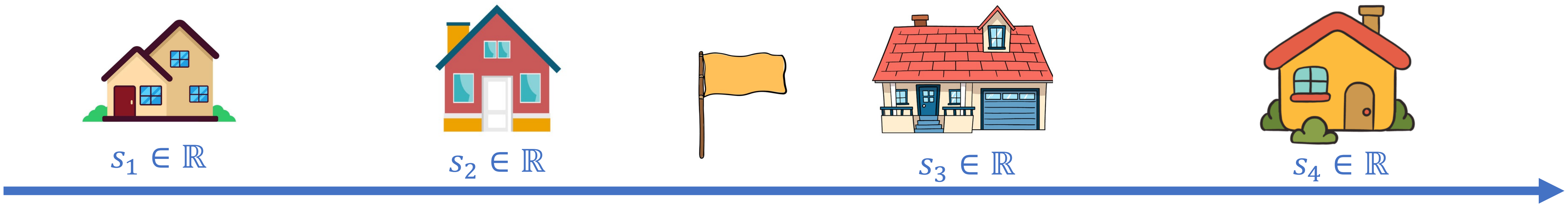
Facility Location

Opening a new study room

Say the principal is going to build a new study room on University Ave. To determine the best locations, we ask all student who live near University Ave to report their address.

We want to find a location x that minimizes

$$\max\left(|x - s_1|,\ |x - s_2|, \dots, |x - s_n|\right).$$

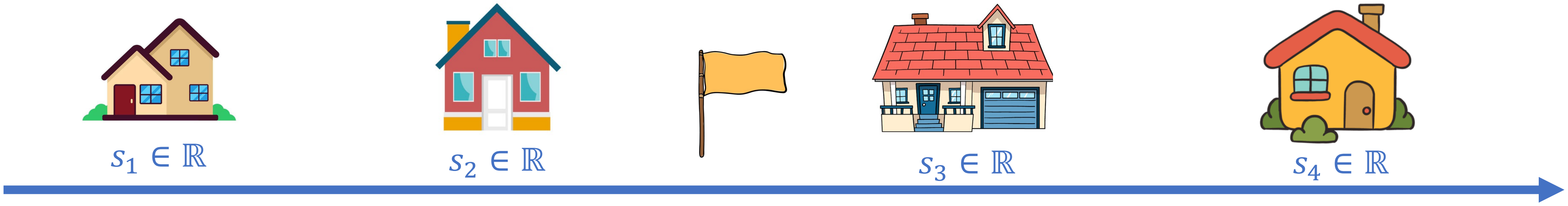


Facility Location

Optimal if all students were truthful:

If we already know the locations, the best place to build the study room is at $\frac{s_1+s_n}{2}$.

- Because it minimizes the distance to furthest student.

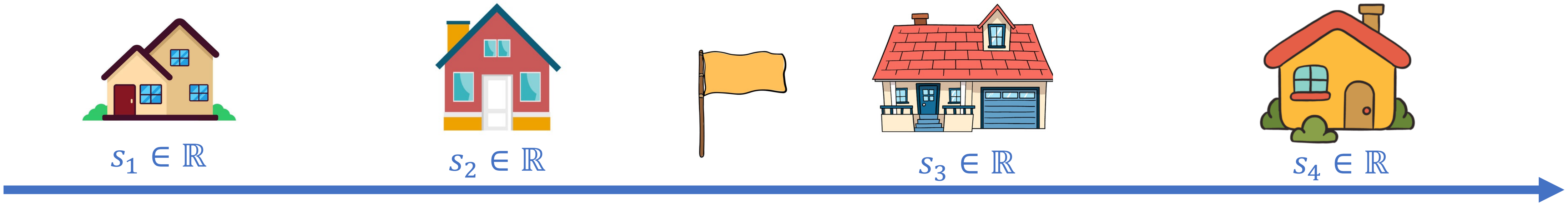


Facility Location

But students may lie….

s_1 could say I actually live in $s_1 - x$.

Then we will just build the library $x/2$ closer to s_1!

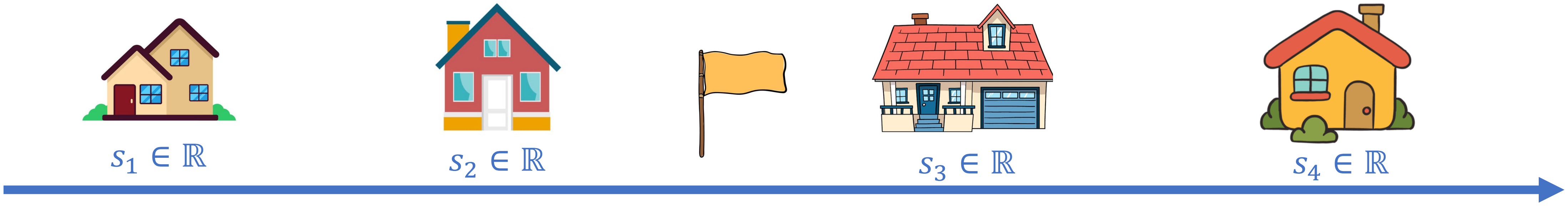


Facility Location

Solution 1:

Just build it the location of a fixed student.

In the worst case, this can be two times longer than the Optimal.



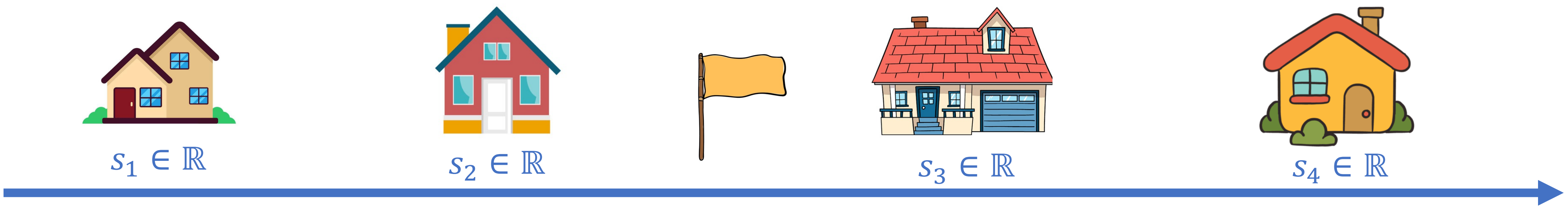
Facility Location

Solution 2:

With 1/4 prob, build it at s_1.

With 1/4 prob, build it at s_n.

With 1/2 prob, build it at $\frac{s_1+s_n}{2}$.



Facility Location

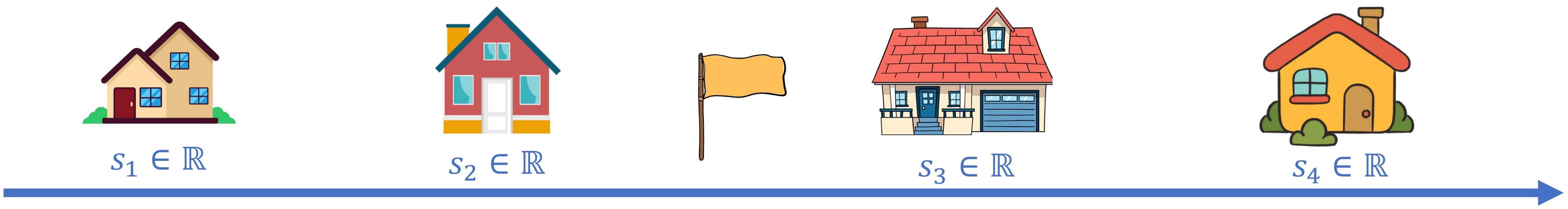
Solution 2: If the first student reports $s_1 - x$ … … .

With 1/4 prob, build it at $s_1 - x$.

With 1/4 prob, build it at s_n.

With 1/2 prob, build it at $\frac{s_1+s_n}{2} - x/2$.

The first student will not lie!



Facility Location

Solution 2:

With 1/4 prob, build it at s_1. -> 2-approx

With 1/4 prob, build it at s_n. -> 2-approx

With 1/2 prob, build it at $\frac{s_1+s_n}{2}$. -> 1-approx. (optimal)

The expected max distance is at most 1.5 times longer than the optimal!

2 * 0.25 + 2* 0.25 + 1 * 0.5 = 1.5