Homework 0

CS 70, Summer 2024

Due by Friday, June 21$^{\text{st}}$ at 11:59 PM

1 Administrivia

(a) `eecs70.org`.

(b) **(i)** After dropping the lowest homework score, the average percent score on the homeworks is

$$\frac{1}{7}(100 + 80 + 84 + 90 + 77 + 87 + 92) \approx 87.14\%.$$

Similarly, after dropping the lowest vitamin score, the average percent score on the vitamins is

$$\frac{1}{6}(100 + 100 + 95 + 100 + 93 + 100) = 98\%$$

The midterm score is lower than the final exam score, so it is worth 30 points and the final exam is worth 110 points. Therefore the student receives approximately

$$0.8714 \cdot 45 + 0.98 \cdot 15 + 0.67 \cdot 30 + 0.73 \cdot 200 = 154.31$$

total points. So the students final percent grade in the class is approximately

$$\frac{154.31}{200} \approx 77.15\%.$$

(ii) Homeworks are released by the end of a week and are due the next week's Friday at 11:59 PM.

(iii) 1 vitamin drop and 1 homework drop.

(iv) Monday, July 15th from 2:00 PM - 3:30 PM.

(v) Wednesday, August 7th from 7:00 PM - 10:00 PM.

2 Collaboration

(a) This is **not** okay. Alice and Bob worked on the homework together, which is perfectly okay, but students should always write up solutions on their own.

(b) This is okay. This falls under students working on a homework together. However, a situation such as this starts to approach academic dishonesty if Dan simply reiterates his solution, or dictates it for someone to copy down. Since Dan is simply explaining his approach and Erin cites Dan, this is fine.

(c) This is okay. Students are allowed to use books or online resources to help solve homework problems as long as they are adequately cited and the material is not copied. It should be noted, though, that heavily relying on external sources to complete the homeworks is generally not a good path to succeeding in this class.

(d) This is **not** okay. Students should never have one another's written work or solutions, and they should not copy one another's solution, even if they cite it.

(e) This is **not** okay. Students should never have one another's written work or solutions. Even though Heidi doesn't copy Irene's solution, this is still a violation of course policy.

(f) This is **not** okay. Solutions to the homework does not count as consulting external resources, and use such solutions constitutes academic dishonesty even if it is cited.

3 Forum

(a) This does not pass the five minute test. Such questions should be reserved for office hours.

(b) Such inquiries should be directed to the course email `cs70-staff@berkeley.edu`.

(c) Thread #3.

(d) Thread #6.

(e) Thread #10.

4 Academic Integrity

> I pledge to uphold the university's honor code: to act with honesty, integrity, and respect for others, including their work. By signing, I ensure that all written homework I submit will be in my own words, that I will acknowledge any collaboration or help received, and that I will neither give nor receive help on any examinations.

5 Sets

(a) $\{n + 0.7 : n \in \mathbb{N}\}$.

(b) **(i)** A_2 is the set of numbers which are divided by two. That's $\{0, 2, 4, 6, 8, 10, \ldots\}$. Those are the even natural numbers.

(ii) $\mathbb{N} \setminus A_2$ is the set of natural numbers without the even numbers. That's the odd natural numbers.

(iii) B is constructed by putting together all the numbers which are divided by 2, all the numbers which are divided by 3, and so on. Note that 1 is not included in B since 1 is not divided by any number. However, every other natural number is included. In particular, for $n > 1$, we have that $n \in A_n$, so $n \in B$. Therefore $\mathbb{N} \setminus B$ is $\mathbb{N} \setminus \{0, 2, 3, 4, 5, \ldots\}$. That's just $\{1\}$.

(iv) The construction of $\mathbb{N} \setminus B$ from before was built by combining a bunch of sets to get a large set and then removing that large set from $\mathbb{N}$.

Here, we'll instead get $\{1\}$ by repeatedly whittling away elements from $\mathbb{N}$. The question helps point us in this direction, since we're only allowed to use set intersection and set differences, which can only result in "smaller" sets.

We'll start with $\mathbb{N} \setminus A_2$. This removes all the even numbers from $\mathbb{N}$, so we've gotten rid of 0 and 2, but still have numbers like $3, 5, 7, 9, \ldots$. Let's start by removing 3 next. We know that $\mathbb{N} \setminus A_3$ doesn't have 3, so we can get rid of 3 by doing $(\mathbb{N} \setminus A_2) \cap (\mathbb{N} \setminus A_3)$. In fact, we have that $\mathbb{N} \setminus A_k$ removes k, so we can remove k by intersecting our set with it. This results in the following intersection:

$$\bigcap_{k=2}^{\infty} (\mathbb{N} \setminus A_k).$$

Note the similarity our equivalence

$$\mathbb{N} \setminus \bigcup_{k=2}^{\infty} A_k = \bigcap_{k=2}^{\infty} (\mathbb{N} \setminus A_k)$$

bears to De Morgan's law. This relationship between set intersection, set union, and set difference holds in general, and is sometimes also known as De Morgan's law.

Note that $\mathbb{N} \setminus B$ can also be viewed as an infinite set difference: $(\ldots(((\mathbb{N} \setminus A_2) \setminus A_3) \setminus A_4)\ldots)$, but this approach is notationally clumsier.

(c) We can first test out a few cases to see. Many simple examples will immediately provide a disproof.

For example, let $A = \{1\}$ and $B = \{2, 3\}$. Then $A \setminus B = \{1\}$, which has cardinality 1. However, $|A| - |B| = 1 - 2 = -1$. So for this example,

$$|A \setminus B| = 1 \neq -1 = |A| - |B|.$$

6 Sums and Products

(a) If we add 5 together twenty times, we're just multiplying it by 20. Similarly, if we multiply 2 together three times, we're just exponentiating by 3. So our expression simplifies to

$$\sum_{i=1}^{20} 5 + \prod_{i=1}^{3} 2 = 20 \cdot 5 + 2^3 = 108.$$

(b) **(i)** $\sum_{i=1}^{5}\left(2c_i+3\right)=\sum_{i=1}^{5}2c_i+\sum_{i=1}^{5}3=2\sum_{i=1}^{5}c_i+5\cdot 3=2\cdot 10+5\cdot 3=35.$

(ii) $\sum_{i=1}^{5}2c_i+3=2\sum_{i=1}^{5}c_i+3=2\cdot 10+3=23.$

(iii) $\sum_{i=1}^{5}\left(2c_i-d_i+3\right)=\sum_{i=1}^{5}(2c_i+3)-\sum_{i=1}^{5}d_i=35-5=30.$

(c) Write out the left-hand side to get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n}x_{ij}=x_{11}+x_{12}+\ldots&+x_{1n}\\ +x_{22}+\ldots&+x_{2n}\\ &\vdots\\ &+x_{nn}.\end{aligned}$$

The new sum adds the same terms together, but it iterates through j first (which corresponds in the diagram above to the columns). The values of j iterate from 1 through n (corresponding to the n columns), so $a=1$ and $b=n$. However, the values of i only go up to j (corresponding to the fact that the j^{th} column has terms only going up to j), so $c=1$ and $d=j$. That is, our equivalent sum with the "order of summation" reversed is

$$\sum_{j=1}^{n}\sum_{i=1}^{j}x_{ij}.$$

Note that this is a discrete version of changing the "order of integration" of a double integral.

(d) **(i)** By the Taylor series expansion of e^x, this is just e^1.

(ii) Write out the first few terms of the summation.

$$\sum_{k=2}^{\infty}\frac{3^k}{k!}=\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\ldots$$

This looks quite a bit like the Taylor series expansion for e^3, except it's missing a few terms. In particular,

$$\begin{aligned}e^3&=\frac{3^0}{0!}+\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}\\ &=1+3+\sum_{k=2}^{\infty}\frac{3^k}{k!}\\ &=4+\sum_{k=2}^{\infty}\frac{3^k}{k!}.\end{aligned}$$

Therefore, by subtracting 4 from both sides,

$$\sum_{k=2}^{\infty}\frac{3^k}{k!}=e^3-4.$$

(iii) This looks like a geometric series, but the $3n$ is in a weird spot. However, note that we can rewrite this as

$$\sum_{n=0}^{\infty}\frac{1}{3^{3n}}=\sum_{n=0}^{\infty}\left(\frac{1}{3^3}\right)^n,$$

which is an infinite geometric series with ratio $1/3^3$. Therefore

$$\sum_{n=0}^{\infty}\frac{1}{3^{3n}}=\frac{1}{1-1/3^3}.$$

(iv) Write out the first few terms of the summation.

$$\sum_{j\in\mathbb{N}} \frac{4^j}{(j+1)!} = \frac{4^0}{1!} + \frac{4^1}{2!} + \frac{4^2}{3!} + \dots$$

Note that this looks quite a bit like the Taylor series expansion for e^4, except the numerator is misaligned with the denominator. To get that back in alignment, we can multiply by a factor of 4:

$$4\sum_{j\in\mathbb{N}} \frac{4^j}{(j+1)!} = 4\left(\frac{4^0}{1!} + \frac{4^1}{2!} + \frac{4^2}{2!} + \dots\right) = \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \dots$$

This looks even more like the Taylor series expansion for e^4, but it's missing the leading $4^0/0! = 1$ term. By the same logic as in **(ii)**, we get that

$$4\sum_{j\in\mathbb{N}} \frac{4^j}{(j+1)!} = e^4 - 1 \iff \sum_{j\in\mathbb{N}} \frac{4^j}{(j+1)!} = \frac{e^4-1}{4}.$$

Note that we can move more quickly towards the answer by manipulating the sum as follows:

$$\sum_{j\in\mathbb{N}} \frac{4^j}{(j+1)!} = \frac{1}{4}\sum_{j\in\mathbb{N}} 4\cdot\frac{4^j}{(j+1)!} = \frac{1}{4}\sum_{j\in\mathbb{N}} \frac{4^{j+1}}{(j+1)!}.$$

Resolving the last sum in the above expression can be done using the same method as part **(ii)**.

(v) This uses many of the same techniques as the previous parts. Re-index the sum. Let $j = k-2$, and the sum becomes

$$\sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^{j+2} = \frac{1}{4^2}\sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^{j} = \frac{1/4^2}{1-1/4},$$

where the last equality follows by the infinite geometric series.