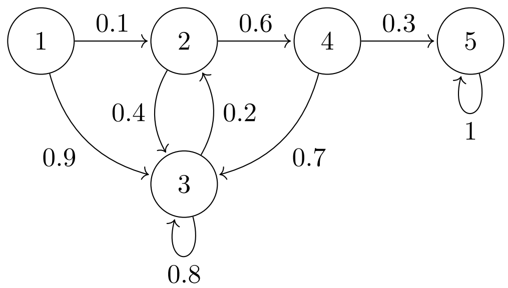
Discussion 6D

CS 70, Summer 2024

1 Random Walk

Consider the following Markov chain with 5 states.



Let X_t be the current state at time t. Suppose initially,

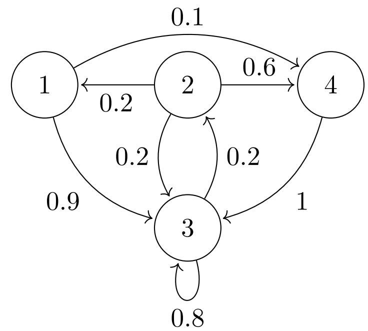
$$\pi_0 = (0.2, 0.3, 0.1, 0.3, 0.1).$$

(a) Without writing down the transition matrix P, calculate what is $\Pr[X_1 = 3]$.

(b) Calculate $\Pr[X_0 = 2 \wedge X_1 = 4 \wedge X_2 = 3 \wedge X_3 = 3 \wedge X_4 = 2 \wedge X_5 = 4]$. Leave your answer as a product.

(c) Without writing down the balance equations, what is the stationary distribution of this Markov chain? Why?

(d) If we modify the Markov chain to the following:



Write down the balance equation (no need to solve it now).

2 Build Your Markov Chain

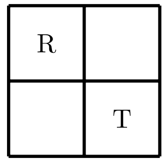
For each of the following random processes, propose a finite-size Markov chain to describe it. You should both draw the diagram, write down the transition matrix P for the Markov chain you build, and the starting distribution π_0.

In the following, H stands for "head" and T stands for "tail".

(a) We repeatedly flip a fair coin and let the sequence $Z_1, Z_2, Z_3 \ldots$ be the result. That is, $Z_i = 1$ if and only if the i-th flip is H (and otherwise $Z_i = 0$). Try to model $Z_1, Z_2, Z_3 \ldots$ as Markov chain. (Let's say $Z_0 = 0$.)

(b) You are trying to use a machine that only works on some days. If on a given day the machine is working, it will break down the next day with probability $0 < b < 1$, and works on the next day $1 - b$. If it is not working on a given day, it will work on the next day with probability $0 < r < 1$, and not work on the next day with probability $1 - r$. Say the initial state of it is uniformly random between these two.

(c) A rabbit is jumping on a two by two grid. Initially, it is at the top left square. At every minute, it jumps to one of the two adjacent square with equal probability. However, there is a trap at the bottom right square. So if the rabbit enters that square, it will never leave that square. See the figure below (R stands for the rabbit and T stands for the trap).



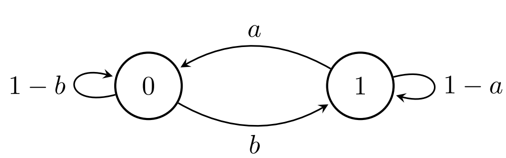
(d) We again repeatedly flip a fair coin. Let $Y_1 = Y_2 = 0$ and for $i \geq 3$, $Y_i = 1$ if and only if the last three flips are HTT (and otherwise $Y_i = 0$). Try to model $Y_1, Y_2, Y_3 \ldots$ as markov chain.

(Hint: Is the process memoryless if we use the value Y_i as our states? What about using the result of the last three flips as our states? Can you use fewer states? There is a solution using four states.)

3 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

- (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
- (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \Pr[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1$ $\forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
- (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$.
- (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.