Discussion 5C

CS 70, Summer 2024

1 Pullout Balls

(a) Let X be the number that you record. Each ball is equally likely to be chosen, so

$$\mathrm{E}[X] = \sum_x x \cdot \Pr[X = x] = 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} = 2.5.$$

As demonstrated here, the expected value of a random variable need not, and often is not, a feasible value of that random variable (there is no outcome ω for which $X(\omega) = 2.5$).

(b) Let Y be the product of two numbers that you pull out. Then

$$\mathrm{E}[Y] = \frac{1}{\binom{4}{2}}(1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4) = \frac{2 + 3 + 4 + 6 + 8 + 12}{6} = \frac{35}{6}.$$

2 Number Game

(a) S is a (discrete) uniform random variable between 0 and 100, so its expectation is $\frac{\sum_{i=0}^{100} i}{101} = 50$.

(b) If $S = s$, we know that V will be uniformly distributed between s and 100. Similar to the previous part, this gives us that $\mathrm{E}[V \mid S = s] = \frac{s+100}{2}$.

(c) With the law of total expectation, we have that

$$\begin{aligned}
\mathrm{E}[V] &= \sum_{s=0}^{100} \mathrm{E}[V \mid S = s] \cdot \Pr[S = s] \\
&= \sum_{s=0}^{100} \frac{s + 100}{2} \cdot \frac{1}{101} \\
&= \frac{1}{202} \left(\sum_{s=0}^{100} s + \sum_{s=0}^{100} 100 \right)
\end{aligned}$$

The first summation comes out to $\frac{100(100+1)}{2} = 50 \cdot 101$; the second summation is just adding 100 to itself 101 times, so it comes out to $100 \cdot 101$. Plugging these values in, we get $\mathrm{E}[V] = 75$.

3 Linearity

(a) Let A_i be the indicator you win the ith time you play game A and B_i be the same for game B. The expected value of A_i and B_i are

$$\begin{aligned}
\mathrm{E}[A_i] &= 1 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = \frac{1}{3}, \\
\mathrm{E}[B_i] &= 1 \cdot \frac{1}{5} + 0 \cdot \frac{4}{5} = \frac{1}{5}.
\end{aligned}$$

Then the expected total number of tickets you receive, by linearity of expectation, is

$$3\mathrm{E}[A_1] + \cdots + 3\mathrm{E}[A_{10}] + 4\mathrm{E}[B_1] + \cdots + 4\mathrm{E}[B_{20}] = 10\left(3 \cdot \frac{1}{3}\right) + 20\left(4 \cdot \frac{1}{5}\right) = 26.$$

Note that $10\left(3 \cdot \frac{1}{3}\right)$ and $20\left(4 \cdot \frac{1}{5}\right)$ matches the expression directly gotten using the expected value of a binomial random variable.

(b) There are $1,000,000 - 4 + 1 = 999,997$ places where "book" can appear, each with a (non-independent) probability of $1/26^4$ of happening. If A is the random variable that tells how many times "book" appears, and A_i is the indicator variable that is 1 if "book" appears starting at the ith letter, then

$$\begin{aligned} \mathrm{E}[A] &= \mathrm{E}[A_1 + \cdots + A_{999,997}] \\ &= \mathrm{E}[A_1] + \cdots + \mathrm{E}[A_{999,997}] \\ &= \frac{999,997}{26^4} \approx 2.19. \end{aligned}$$

4 Number of Ones

Solution using the self-referencing trick. Let X be the number of ones we see until we see a 6. Let $r_1, r_2, \ldots$ be each roll of the die.

We know that

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}[X \mid r_1 = 1] \cdot \Pr[r_1 = 1] + \mathrm{E}[X \mid r_1 = 6] \cdot \Pr[r_1 = 6] + \mathrm{E}[X \mid r_1 \notin \{3,6\}] \cdot \Pr[r_1 \notin \{3,6\}] \\ &= \mathrm{E}[X \mid r_1 = 1] \cdot \frac{1}{6} + \mathrm{E}[X \mid r_1 = 6] \cdot \frac{1}{6} + \mathrm{E}[X \mid r_1 \notin \{3,6\}] \cdot \frac{2}{3} \cdot \mathrm{E}[X|r_1 \notin \{3,6\}] \end{aligned}$$

Suppose $r_1 = 1$, we already see an 1 and we will keep rolling. The expected number of more 1's we will see is just $\mathrm{E}[X]$. So $\mathrm{E}[X \mid r_1 = 1] = \mathrm{E}[X] + 1$. If $r_1 = 6$, we will immediately stop. So $\mathrm{E}[X \mid r_1 = 6] = 0$. Finally, if r_1 is not one of these two, we basically wasted one roll, and the expected number of 1's we will see is still $\mathrm{E}[X]$. So $\mathrm{E}[X \mid r_1 \notin \{1,6\}] = \mathrm{E}[X]$.

As a result, $\mathrm{E}[X] = (1 + \mathrm{E}[X]) \cdot \frac{1}{6} + \mathrm{E}[X] \cdot \frac{2}{3}$. We can solve the equation and get $\mathrm{E}[X] = 1$.

Solution using law of total probability. Below is an alternative solution which is kind of a brute force calculation. Just by looking at its length, one could already appreciate the elegance of the self-referencing trick is.

Let Y be the number of ones we see. Let N be the number of rolls we take until we get a 6.

Let us first compute $\mathrm{E}[Y \mid N = k]$. We know that in each of our $k - 1$ rolls before the kth, we necessarily roll a number in $\{1, 2, 3, 4, 5\}$. Thus, we have a $1/5$ chance of getting a one in each of these $k - 1$ previous rolls, giving

$$\mathrm{E}[Y \mid N = k] = \frac{1}{5}(k - 1).$$

If this is confusing, we can write Y as a sum of indicator variables, $Y = Y_1 + Y_2 + \cdots + Y_k$, where Y_i is 1 if we see a one on the ith roll. This means that by linearity of expectation,

$$\mathrm{E}[Y \mid N = k] = \mathrm{E}[Y_1 \mid N = k] + \mathrm{E}[Y_2 \mid N = k] + \cdots + \mathrm{E}[Y_k \mid N = k].$$

We know that on the kth roll, we must roll a 6, so $\mathrm{E}[Y_k] = 0$. Further, by symmetry, each term in this summation has the same value; this means that we have

$$\begin{aligned} \mathrm{E}[Y_1 \mid N = k] + \mathrm{E}[Y_2 \mid N = k] + \cdots + \mathrm{E}[Y_{k-1} \mid N = k] &= (k - 1)\mathrm{E}[Y_1 \mid N = k] \\ &= (k - 1) \Pr[Y_1 = 1 \mid N = k] \\ &= (k - 1)\frac{1}{5}. \end{aligned}$$

Using the law of total expectation, we now have

$$\begin{aligned} \mathrm{E}[Y] &= \sum_{k=1}^{\infty} \mathrm{E}[Y \mid N = k] \Pr[N = k] && \text{(total expectation)} \\ &= \sum_{k=1}^{\infty} \frac{1}{5}(k - 1) \Pr[N = k] \\ &= \mathrm{E}\left[\frac{1}{5}(N - 1)\right] \\ &= \frac{1}{5}(\mathrm{E}[N] - 1) && \text{(linearity)} \end{aligned}$$

Note the expected number of rolls until we roll a 6 is $\mathrm{E}[N] = 6$:

$$= \frac{1}{5}(6 - 1) = 1$$

Alternatively, we can use iterated expectation, along with the fact that $\mathrm{E}[Y \mid N] = \frac{1}{5}(N - 1)$, to give

$$\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}[Y \mid N]] \\
&= \mathrm{E}\frac{1}{5}(N - 1) \\
&= \frac{1}{5}(\mathrm{E}[N] - 1) \\
&= \frac{1}{5}(6 - 1) = 1
\end{aligned}$$