Discussion 4B

CS 70, Summer 2024

1 Counting Practice

Count each of the following quantities.

(a) For $n \in \mathbb{N}$, find the number of subsets of $\{0, 1, 2, 3, \ldots, n\}$.

(b) For a positive integer $k \in \mathbb{Z}^+$, a k-clique is a set of vertices in a graph which are all adjacent to one another. Two cliques are considered the same if they contain the same vertices.

For $n, k \in \mathbb{Z}^+$ two positive integers, find the number of possible k-cliques in a graph with n vertices.

(c) Note that 1-cliques and 2-cliques are rather boring, since they're just vertices and edges. We'll call such cliques *trivial cliques.*

Find the number of possible nontrivial cliques in a graph with n vertices.

(d) For $n \in \mathbb{Z}^+$, find the number of nonnegative integer solutions to $x + y + z = n$.

(e) For $n \in \mathbb{Z}^+$, find the number of non-increasing n-digit sequences.

(f) For $n \in \{1, \ldots, 10\}$, find the number of strictly decreasing n-digit sequences.

(g) A small company has $n > 4$ workers. Each worker's birthday is in winter, spring, summer, or fall. Find the number of ways that there can be no birthdays in winter.

(h) Find the number of ways that there could be at least one season in which there are no birthdays.

2 Casting Counting

Alyssa has been working on a musical. She needs to select a cast, crew, and directing team to staff her musical.

(a) Alyssa has received applications from $2n$ directors for her 2 director positions. Some of the directors are experienced, while the others are inexperienced.

Use this setting to provide a combinatorial proof of the following identity.

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Alyssa needs to select a cast of k people from the n cast applications she receives. She's familiar with one of the applicants, so she looks at their application first.

Use this setting to provide a combinatorial proof of the following identity.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) Alyssa needs to hire the actors for her musical. She knows she'll cast only one actor for the lead role, but she doesn't know how many of the n actors she'd like to cast.

Use this setting to provide a combinatorial proof of the following identity.

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

(d) Provide a combinatorial proof of the following identity.

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

It'll be helpful to come up with a setting, or to use the same setting from the previous parts.

3 A Totient Identity

For each positive integer $j \in \mathbb{Z}^+$, let

$$S_j = \left\{k \in \mathbb{N}^+ : k \leq j \text{ and } \gcd(j, k) = 1\right\}$$

be the set of positive integers up to j which are coprime with j. The *totient function* $\varphi : \mathbb{N}^+ \to \mathbb{N}^+$ is defined as

$$\varphi(n) = |S_n|$$

for each $n \in \mathbb{N}^+$. That is, for any $n \in \mathbb{N}^+$, $\varphi(n)$ is the count of positive integers up to n which are relatively prime to n.

Prove that for any positive natural $n \in \mathbb{N}^+$,

$$\sum_{d|n} \varphi(d) = n.$$