Discussion 1C

CS 70, Summer 2024

1 The Triangle Inequality

You may remember from a previous math class the *triangle inequality*, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

In this question, we will generalize the triangle inequality using mathematical induction to prove that

$$|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|.$$

(a) State the base case.

(b) State the induction hypothesis.

(c) Show the induction step.

2 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that for any positive integer n, we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

for some $k \in \mathbb{N}$ and $c_i \in \{0, 1\}$ for all $i \leq k$.

(*Hint: in the induction step, consider the case where $n + 1$ is even and the case where $n + 1$ is odd.*)

3 Stones

Charlize and Beomgyu like play a game involving a pile of n stones. They alternate taking turns removing stones from the pile. On each turn, the player removes either one or two stones from the pile. The last player to make a turn loses.

Charlize makes the first move.

(a) A common technique for problem-solving and proof-writing is to work with small examples. In this case, that's when n is small.

Determine who would win if there are $n = 1$ stones in the pile. Do the same for when there are $n = 2, 3, 4$ stones.

(b) After working through a few examples, we try to catch any patterns to make a conjecture.

For the three cases where $n = 3k + 1$, $n = 3k + 2$, and $n = 3k + 3$, conjecture who you think would win the game.

(c) Prove your conjecture.

4 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We are interested in proving the claim that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Consider a proof by induction with the induction hypothesis $a_n \leq 3^{(2^n)}$. Work through the details of the induction proof to demonstrate that this induction hypothesis will not work.

(b) Prove instead that $a_n \leq 3^{(2^n-1)}$.

(c) Use the result from part **(b)** to prove the original claim.