Discussion 1C

CS 70, Summer 2024

1 The Triangle Inequality

(a) **Base case**. By the triangle inequality, $|x_1 + x_2| \leq |x_1| + |x_2|$.

(b) **Induction hypothesis**. For some $n \geq 2$, suppose that

$$|x_1 + x_2 + \ldots + x_{n-1}| \leq |x_1| + |x_2| + \ldots + |x_n|$$

holds.

(c) **Induction step**. Let $x_1, \ldots, x_{n+1} \in \mathbb{R}$. Then

$$\begin{aligned} |x_1 + \ldots + x_{n+1}| &= |(x_1 + \ldots + x_n) + x_{n+1}| \\ &\leq |x_1 + \ldots + x_n| + |x_{n+1}| && \text{(triangle inequality)} \\ &\leq |x_1| + \ldots + |x_n| + |x_{n+1}|. && \text{(induction hypothesis)} \end{aligned}$$

2 Binary Numbers

By strong induction on n.

Base case. $n = 1 = 1 \cdot 2^0$.

Induction case.

Induction hypothesis. Suppose that for all $1 \leq m \leq n$, we can write k in binary.

Induction step. Consider $n+1$. We consider the two cases where $n+1$ is even and $n+1$ is odd.

(1) $n+1$ is even. Then $(n+1)/2 \leq n$ is an integer. By the induction hypothesis, we can write $(n+1)/2$ in binary. That is, there exist $b_0, \ldots, b_k \in \{0, 1\}$ such that

$$(n+1)/2 = b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + \ldots + b_1 \cdot 2^1 + b_0 \cdot 2^0.$$

Then

$$\begin{aligned} (n+1)/2 &= b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + \ldots + b_1 \cdot 2^1 + c_0 \cdot 2^0 \\ n+1 &= 2\left(b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + \ldots + b_1 \cdot 2^1 + b_0 \cdot 2^0\right) \\ &= b_k \cdot 2^{k+1} + b_{k-1} \cdot 2^k + \ldots + b_1 \cdot 2^2 + b_0 \cdot 2^1 \\ &= c_{k+1} \cdot 2^{k+1} + c_k \cdot 2^k + \ldots + c_2 \cdot 2^2 + c_1 \cdot 2^1 + c_0 \cdot 2^0, \end{aligned}$$

where $c_j = d_{j-1}$ for $1 \leq j \leq k+1$ and $c_0 = 0$.

(2) $n+1$ is odd. Then n is even and by the induction hypothesis, there exist $b_0, \ldots, b_k \in \{0, 1\}$ such that

$$n = b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + \ldots + b_1 \cdot 2^1 + b_0 \cdot 2^0.$$

We claim that $b_0 = 0$. In particular,

$$\begin{aligned} n &= b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + \ldots + b_1 \cdot 2^1 + b_0 \cdot 2^0 \\ &= 2\left(b_k \cdot 2^{k-1} + b_{k-1} \cdot 2^{k-2} + \ldots + b_1 \cdot 2^0\right) + b_0. \end{aligned}$$

If $b_0 = 1$, then n is of the form $2\ell + 1$ for some $\ell \in \mathbb{Z}$, and is therefore odd. This is a contradiction, so it must be that $b_0 = 0$. Then

$$\begin{aligned} n &= b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + \ldots + b_1 \cdot 2^1 + 0 \cdot 2^0 \\ n+1 &= b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + \ldots + b_1 \cdot 2^1 + 1 \cdot 2^0 \\ &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0, \end{aligned}$$

where $c_j = b_j$ for $1 \leq j \leq k$ and $c_0 = 1$.

By the principle of mathematical induction, we have shown the claim.

3 Stones

(a) When $n = 1$ or $n = 4$, Beomgyu, the next player, wins. When $n = 2$ or $n = 3$, Charlize, the current player wins.

(b) We conjecture that when $n = 3k + 1$, Beomgyu, the next player, wins, and that when $n = 3k + 2$ or $n = 3k + 3$, Charlize, the current player, wins.

(c) We will prove the stronger claim that when $n = 3k+2$ or $n = 3k+3$, the current player wins, and that when $n = 3k+1$, the next player wins. We use strong induction.

Base case. $n = 1$. By part **(a)**, the next player, Beomgyu, wins.

Induction case.

Induction hypothesis. Suppose that for each $m \in \{1, \ldots, n\}$, the claim is true for a pile with m stones.

Induction step. Consider a pile with $n + 1$ stones. There are three cases. Either $n + 1 = 3k + 1$, $n + 1 = 3k + 2$, or $n + 1 = 3k + 3$.

(1) $n + 1 = 3k + 1$. If the current player removes one stone from the pile, there will remain $n = 3k = 3(k - 1) + 3$ stones, and the opponent becomes the current player. By the induction hypothesis, the opponent wins.

If the current player removes two stones from the pile, there will remain $n = 3k - 1 = 3(k - 1) + 2$ stones, and the opponent becomes the current player. By the induction hypothesis, the opponent wins.

No matter what the current player does, their opponent wins. So the next player wins.

(2) $n + 1 = 3k + 2$. Then the current player can remove one stone from the pile to get a pile with $3k + 1$ stones, and the opponent becomes the current player. By the induction hypothesis, the opponent's next player (who is the current player) wins.

(3) $n + 1 = 3k + 3$. Then the current player can remove two stones from the pile to get a pile with $3k + 1$ stones. By the previous case, the current player wins.

In each case, the claim holds, so it holds in general.

By the principle of mathematical induction, we have proved our conjecture.

4 Make It Stronger

(a) We attempt a proof by induction.

Base case. For $n = 1$, $a_1 = 1 \leq 9 = 3^{(2^1)}$.

Induction case.

Induction hypothesis. For some $n \geq 1$, suppose that $a_n \leq 3^{(2^n)}$.

Induction step. Consider a_{n+1}.

$$\begin{aligned}
a_{n+1} &= 3a_n^2 \\
&\leq 3\left(3^{(2^n)}\right)^2 && \text{(induction hypothesis)} \\
&= 3\left(3^{(2\cdot 2^n)}\right) \\
&= 3\left(3^{(2^{n+1})}\right) \\
&= 3^{(2^{n+1}+1)} \\
&\not\leq 3^{(2^{n+1})}.
\end{aligned}$$

(b) By induction.

Base case. For $n = 1$, $a_1 = 1 \leq 9 = 3^{(2^1)}$.

Induction case.

Induction hypothesis. For some $n \geq 1$, suppose that $a_n \leq 3^{(2^n-1)}$.

Induction step. Consider a_{n+1}.

$$\begin{aligned}
a_{n+1} &= 3a_n^2 \\
&\leq 3\left(3^{(2^n-1)}\right)^2 && \text{(induction hypothesis)} \\
&= 3\left(3^{(2\cdot 2^n-2)}\right) \\
&= 3\left(3^{(2^{n+1}-2)}\right) \\
&= 3^{(2^{n+1}-1)}.
\end{aligned}$$

By the principle of mathematical induction, we have shown that for every natural number $n \geq 1$, $a_n \leq 3^{(2^n-1)}$.

(c) For every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and so $3^{(2^n-1)} \leq 3^{(2^n)}$.

By part **(b)**, $a_n \leq 3^{(2^n-1)} \leq 3^{(2^n)}$, which is the claim we wanted to show in **(a)**.